Fuzzy Conditional Reliability for Some Continuous Distributions

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Abstract

Fuzzy set theory has become a useful tool in the mathematical modeling of problem in operation research, including reliability engineering and applied recently in various fields. The problem of reliability with fuzziness is discussed by means of fuzzy mathematics. In the world, there exists much fuzzy knowledge; knowledge that is vague, imprecise, uncertain, ambiguous, inexact or probabilistic in natural. Human thinking and reasoning frequently involve fuzzy information, originating from inherently inexact human concepts. Fuzzy sets have been able provide solutions to many real world problems. Suppose that an item has survived to time a, where T is the total time, then U = T - a is the future life random variable. The conditional reliability function defined by:

\[ R_U(t | a) = Pr(\text{item survives a further time } t \mid \text{survives to } a) \]

Fuzzy reliability is the probability of a device performing its purpose in varying degrees of success for the period of time intended under operating conditions encountered. The aim of this paper find the fuzzy condition reliability of a some continuous distribution where the random variable independent and dependent.

Key words: Fuzzy set, Fuzzy reliability, Exponential distribution, Weibull distribution Fuzzy conditional

Introduction

In the world, there exists much fuzzy knowledge; knowledge that is vague, imprecise, uncertain, ambiguous, inexact or probabilistic in natural. Human thinking and reasoning frequently involve fuzzy information, originating from inherently inexact human concepts. Fuzzy sets have been able provide solutions to many real world problems. The concepts of fuzzy set have been introduced by Lotfi A. Zadeh (1965) as an extension of classical notion of set and its developed to probability measures of fuzzy event by Lotfi A. Zadeh (1968) and M. Sugeno (1974) Then developed the concepts of reliability which is defined in terms of probability, probabilistic parameters such as random variables, density function, distribution function and system reliability by Peter Harding and Associates Pty ltd (1996). The relationship between fuzzy and reliability has been proposed and developed by several authors (Cai et al., 1991; Chen, 1994; Cai et al., 1995). Then using the membership functions and probability measures of fuzzy set by Nozer D. Singh purwalla and Jane M. Brooker (2004) Fuzzy reliability is the probability of a device performing its purpose in varying degrees of success for the period of time intended under operating conditions encountered. The aim of this paper find the fuzzy condition reliability of a some continuous distribution where the random variable independent and dependent.
2. Fuzzy Conditional Reliability

Suppose that an item has survived to time $a$, where $T$ is the total time, then $U = T - a$ is the future life random variable. The conditional reliability function defined by:

$$R_U \left( t | a \right) = \Pr(\text{item survives a further time } t | \text{survives to } a)$$

$$= \Pr(\text{U} > t | T > a) \quad u > 0$$

$$= \Pr(\text{T} > a + t | T > a) \quad u > 0$$

$$= \frac{\Pr(\text{T} > a + t)}{\Pr(\text{T} > a)}$$

$$= \frac{R(a + t)}{R(a)}$$

If the sign $A$ denotes that a device performs its purpose adequately and the sign $\hat{A}_i$ denotes one performance subsets, then in terms of the definition of fuzzy conditional probability, we have

$$P(\text{A} \land \hat{A}_i) = \Pr(\hat{A}_i | A) P(\text{P(A)})$$

Substituting Eq. (2) in to Eq. (1), we obtain

$$\hat{R} = P(\hat{A}_i | A) R$$

Suppose that $\mu_{\hat{A}_i}(R)$ is the degree of membership of $R$ in $\hat{A}_i$ and substitute $\mu_{\hat{A}_i}(R)$ for $P(\hat{A}_i | A)$, then

$$\hat{R} = \mu_{\hat{A}_i}(R) R$$

According to general reliability theory we have,

$$R = \int_{t}^{\infty} f(t) \, dt$$

\ldots \ldots
Substituting it in to Eq. (4), we obtain

\begin{equation}
\overline{R} = \mu \int_{t}^{\infty} f(t) \:\! dt \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots '\n

3. Some Continuous Distribution
3.1 Exponential Distribution

The exponential distribution is a commonly used distribution in reliability engineering. Mathematically, it is a fairly simple distribution, which many times lead to its use in inappropriate situations. The exponential distribution is used to model the behavior of units that have a constant failure rate (or units that do not degrade with time or wear out).

The two-parameter exponential pdf is given by:

\[ f(t) = \lambda e^{-\lambda(t-\gamma)} \quad t > 0, \: \lambda > 0, \: \gamma > 0 \]

where \( \gamma \) is the location parameter, and \( \lambda \) is the scale parameter.

3.1.1 Exponential Statistical Properties

The mean \( E(T) \) is given by:

\[ E(T) = \int_{\gamma}^{\infty} tf(t) \: dt \]

\[ E(T) = \frac{1}{\lambda} \]

and the variance \( \text{Var}(T) \) is:

\[ \text{Var}(T) = \frac{1}{\lambda^2} \]

3.1.2 The Exponential Reliability Function

The equation for the two-parameter exponential cumulative density function is given by:

\[ F(t) = \int_{0}^{t} f(t) \: dt \]

\[ F(t) = 1 - e^{-\lambda(t-\gamma)} \]

The reliability function for two-parameter exponential distribution is given by:

\[ R(t) = 1 - F(t) \]

\[ R(t) = e^{-\lambda(t-\gamma)} \]

The hazard function (failure rate function) of exponential distribution is given by:

\[ h(t) = \frac{f(t)}{R(t)} = e^{-\lambda(t-\gamma)} \]

\[ h(t) = \lambda \]

3.2 Weibull Distribution

Weibull distribution is one of the most widely used lifetime distributions in reliability engineering. It is a versatile distribution that can take on the characteristics of other
types of distributions, based on the value of the shape parameter, $\beta$.

The two-parameter Weibull pdf is given by:

$$f(t) = \frac{\beta}{\gamma} \left(\frac{t}{\gamma}\right)^{\beta-1} e^{-\left(\frac{t}{\gamma}\right)^\beta} \quad t \geq 0, \quad \beta > 0, \quad \gamma > 0$$

where $\beta$ is the shape parameter and $\gamma$ is the scale parameter.

### 3.2.1 Weibull Statistical Properties

The mean $E(T)$ is given by:

$$E(T) = \gamma \Gamma\left(\frac{1}{\beta} + 1\right)$$

and the variance $\text{Var} (T)$ is:

$$\text{Var} (T) = \gamma^2 \Gamma\left(1 + \frac{2}{\beta}\right) - (E(T))^2$$

### 3.2.2 The Weibull Reliability Function

The equation for the two-parameter Weibull cumulative density function is given by:

$$F(t) = \int_0^t f(t) dt = 1 - e^{-\left(\frac{t}{\gamma}\right)^\beta}$$

The reliability function for two-parameter Weibull distribution is given by:

$$R(t) = 1 - F(t) = e^{-\left(\frac{t}{\gamma}\right)^\beta}$$

The hazard function (failure rate function) of Weibull distribution is given by:

$$h(t) = \frac{f(t)}{R(t)} = \frac{\beta}{\gamma} \left(\frac{t}{\gamma}\right)^{\beta-1} e^{-\left(\frac{t}{\gamma}\right)^\beta} = \left(\frac{t}{\gamma}\right)^{\beta-1}$$

[Karpisek Zdenek, Stepanek Petr, Jurak Petr, 2010]

### 4. Fuzzy Conditional Reliability for Independ and Depended

In this paper, we discussed the fuzzy conditional reliability for some continuous distribution where random variables are independent and dependent.

#### 4.1 When they have the same independent distribution

#### 4.1.1 Exponential distribution
R(t) = e^{-\lambda(t-\gamma)}
R(t') = e^{-\lambda(t'-\gamma)}

\begin{align*}
R(t \mid t') &= \frac{R(t) \ast R(t')}{R(t')}
&= \frac{e^{-\lambda(t-\gamma)} \ast e^{-\lambda(t'-\gamma)}}{e^{-\lambda(t-\gamma)}}
&= e^{-\lambda(t'-\gamma)}
\end{align*}

Where \( t \) and \( t' \) are random variables of survives times, that have Exponential distribution.

4.1.2 Weibull Distribution

\begin{align*}
R(t^\circ) &= e^{-\left(\frac{t^\circ}{\gamma}\right)^\beta}
R(t^*) &= e^{-\left(\frac{t^*}{\gamma}\right)^\beta}

R(t \mid t^*) &= \frac{R(t^\circ) \ast R(t^*)}{R(t^*)}
&= \frac{e^{-\lambda(t^\circ-\gamma)} \ast e^{-\lambda(t^*-\gamma)}}{e^{-\lambda(t^* \gamma)}}
&= e^{-\lambda(t^\circ-\gamma)}
\end{align*}

Where \( t^\circ \) and \( t^* \) are random variables of survives times, that have Weibull distribution.

4.2 When they have the different independent distribution

For Exponential distribution the reliability function is

\begin{align*}
R(t) &= e^{-\lambda(t-\gamma)}
\end{align*}

And for Weibull distribution the reliability is

\begin{align*}
R(t^*) &= e^{-\left(\frac{t^*}{\gamma}\right)^\beta}
\end{align*}

Then,

\begin{align*}
R(t \mid t^*) &= \frac{R(t) \ast R(t^*)}{R(t^*)}
&= \frac{e^{-\lambda(t-\gamma)} \ast e^{-\left(\frac{t^*}{\gamma}\right)^\beta}}{e^{-\left(\frac{t^*}{\gamma}\right)^\beta}}
&= e^{-\lambda(t-\gamma)}
\end{align*}
4.3 When they have the same dependent distribution

4.3.1 Exponential distribution

The membership of Exponential distribution is given by:

$$\mu(t) = \frac{1}{\frac{\pi t}{\sqrt{6 m}}}$$

Since,

$$\bar{R} = \mu_{\lambda_0}(R) \ R$$

Then,

$$R(\mathbf{t} | t') = \frac{\bar{R}}{\bar{R}}$$

$$= \frac{2 \left(1 + \frac{\pi t}{\sqrt{6 m}}\right)^{-1} \cdot e^{-\lambda (t'-\gamma)}}{e^{-\lambda (t' - \gamma)}}$$

Where $(\mathbf{t} | t') = t''$.

4.3.2 Weibull distribution

The membership of Weibull distribution is given by:

$$\mu(t) = \frac{1}{(1 + kt^2)}$$

Where $k$ is parameter

Since,

$$\bar{R} = \mu_{\lambda_0}(R) \ R$$

$$= \frac{1}{(1 + kt^2)} \cdot e^{-\left(\frac{t}{\gamma}\right)^\beta}$$
Then,

\[
R(t|t^*) = \frac{\text{R}}{R} \left(1 + k((t^*)^\beta - t^\beta)\left(1 + k((t^*)^\beta - t^\beta)^2\right)\right)
\]

Where \( t^* \) is the parameter��

From the above work we conclusion that in the Exponential distribution and Weibull distribution the fuzzy conditional reliability gives same distributions when they are independent distributions, and gives the membership function of Exponential and Weibull distributions when they are depended distribution.

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