On The Mixed $r$ Th Modulus of Smoothness

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Abstract
We introduce the mixed 1st modulus of smoothness of functions in $L_p(X)$, for $p<1$, for Peano continuum $X$. Then we define a mixed $r$th modulus of smoothness of functions in $L_p(X)$. Some properties and direct theorems for these moduli of smoothness are proved.

Key words. Mixed modulus of smoothness. Degree of approximation. Direct theorem.

1. The first mixed modulus of smoothness
In our work we use $X$ as a compact space under the metric $d_X$ also we use $L_p(X)$, $p<1$, the space of all functions $f: X \rightarrow \mathbb{R}$ satisfying $\|f\|_p = \left( \int_X |f|^p \right)^{1/p} < \infty$. We mean by the Peano continuum, any locally connected, compact metric space.

Let $X$ and $Y$ be two compact spaces under the matrices $d_X$ and $d_Y$ respectively, and if $g$ a real function on $X \times Y$, it mean in $L_p(X \times Y)$. Then we define a version of mixed modulus of smoothness of first order as

$$\omega_{1,1} (g, \sigma_1, \sigma_2) = \sup_{d_{X}(x_1, x_2) \leq \sigma_1} \sup_{d_{Y}(w_1, w_2) \leq \sigma_2} \left\| g(x_1, w_1) - g(x_2, w_2) - g(z_1, w_1) + g(z_2, w_2) \right\|_p$$

Let us collect some properties of the first mixed modulus of smoothness by the following theorem, of easy direct proof.

Theorem 1.1. If $g \in L_p(X \times Y)$, $p < 1$ then
1.1.1. $\omega_{1,1} (g, 0, 0) = 0$
1.1.2. $\omega_{1,1} (g, \sigma_1, \sigma_2)$ is monotone function of $(\sigma_1, \sigma_2)$
1.1.3. $\omega_{1,1} (f, \lambda \sigma_1, \lambda \sigma_2) \leq c(p) \lambda \lambda_2 \omega_{1/1} (f, \sigma_1, \sigma_2)$
1.1.4.

2. $r$th order mixed modulus for measuring smoothness
In this section we will define the mixed $r$th modulus of smoothness and introduce some theorems as applications of it.

If $f$ is a real function on $X \times Y$ belongs to $L_p(X \times Y)$ define the mixed $r$th modulus of smoothness, for $r \geq 2$ as

$$\omega_{r,r} (f, \delta_1, \delta_2) = \sup_{0 < h_1 < \delta_1, 0 < h_2 < \delta_2} \left\| \sum_{i=0}^{T} (-1)^{r-i} \left( f(x, y - \frac{r h_1}{2} + i h_1) - f(x - \frac{r h_2}{2} i h_2, y) \right) \right\|_p$$

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\( \delta_1, \delta_2 > 0, \) when \( x \pm \frac{nh}{2} \in X , = 1,2. \)

In the following theorem let us collect some properties of our mixed modulus of smoothness

**Theorem 2.1.** Let \( f \in L^p(X) \) \( p < 1, \) where \( X \) is a Peano cotinum metric space under the metric \( d, \) then

1. \( \omega_{r,r} (f, 0, 0)_p = 0 \)
2. \( \omega_{r,r} (f, \delta_1, \delta_2)_p \geq 0 \)
3. \( \omega_{r,r} (f, \delta_1, \delta_2)_p \leq \omega_{r,r} (f, \delta_1', \delta_2')_p, \)

when \( \delta_1 \leq \delta_1' , \delta_2 \leq \delta_2'. \)

1.4 \( \omega_{r,r} (f, \lambda_1 \delta_1, \lambda_2 \delta_1)_p \leq c (\lambda_1 \lambda_2 \omega_{r,r} (f, \delta_1, \delta_2)_p) \)

1.5 \( \omega_{r,r} (f, \delta_1 + \delta_2, \lambda_1 + \lambda_2)_p \leq c (\omega_{r,r} (f, \delta_1, \lambda_1)_p + \omega_{r,r} (f, \delta_2, \lambda_2)_p) \)

**proof:** the proofs of 2.1.1 and 2.1.2, are direct. Now let us prove 2.1.3. Let \( \delta > 0, \) we have \( \delta_2 \leq \lambda \delta_2, \) by a result from functional analysis there exists \( \tilde{h}_2 \) satisfy \( C_1 h_2 \leq \tilde{h}_2 \leq C_2 h_2, \) \( C_1 , C_2 \) are positive constants. Since \( X \) is a compact space, using a version of Hilbert theorem we obtain that there exists a shortest arc \( \Gamma \) connecting any two points from

\[ \{ \{ t_i \} | t^r = 0 = \{ X - \frac{r h_2}{2} + i h_2 \} \} \]

and \( h_2 = d(t_i, t_{i+1}), i = 0, 1, \ldots., r \) and \( h_2 \leq d(t_i, t_{i+1}). \)

Since \( X \) is convex metric space we obtain that length \( \Gamma = h_2 \leq C_2 h_2 \leq C_2 \lambda_2 \delta_2 \leq C_1 \delta_2. \)

**Proof of 2.1.4.** If \( \delta_1 = \delta_2 = 0 \) the proof is trivial, so let us assume \( \delta_1 , \delta_2 > 0, \) and let \( (x_1, y_1) \) and \( (x_2, y_2) \) are two points in \( X \times X \) \( d(x_1, x_2) \leq \lambda_1 \delta_1 \) and \( d(y_1, y_2) \leq \lambda_2 \delta_2. \)

From analysis we can find metrices \( f_x \) and \( f_y \) on \( X \) and \( Y \) respectively equivalent to \( d_x \) and \( d_y \) respectively. Because of the compactness of \( X \) and \( Y, \) from analysis there is an arc \( \Gamma_1 \) connecting \( x_1 \) and \( y_1, \) also there is an arc \( \Gamma_2 \) connecting \( x_2 \) and \( y_2, \) and \( f_x (x_1, x_2) \) is the length of the arc \( \Gamma_1, \) and \( f_y (y_1, y_2) \) is the length of the arc \( \Gamma_2. \)

Then the length of \( \Gamma_1 = f_x (x_1, x_2) \leq c d_x (x_1, x_2) \leq \lambda_1 \delta_1, \) also the length of \( \Gamma_2 = f_y (y_1, y_2) \leq c d_y (y_1, y_2) \leq \lambda_2 \delta_2. \)

Let \( \ell_i = \frac{i}{n}, i = 0, 1, 2, 3, \ldots., n, \) we can find a parametrization

\[ \psi_1 \) with \( z_i = \psi_1 (\ell_i), \) and \( \psi_2 \) with \( w_i = \psi_2 (\ell_i), \) and \]

\[ d_x (\psi_1 (\ell_i), \psi_1 (\ell_{i+1}) \leq c f_x (\psi_1, \ell_i), \psi (\ell_{i+1}) \leq c. \]

The length of \( \Gamma_1 \) connecting \( z_i \) and \( z_{i+1} \)

\[ = c (\ell_{i+1} - \ell_i) \]

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Then

\[ d_x (z_i, z_{i+1}) \leq \delta_1 \text{ and } d_y (w_i, w_{i+1}) \leq \delta_2, \text{ for } i = 0, 1, 2, n. \]

If we assume \( \lambda_1, \lambda_2 = cn, \) this leads to

\[ \| f(x_1, y_1) - f(x_2, y_2) - f(x_2, y_1) + f(x_2, y_2) \|_p \leq \sum_{i=0}^{n-1} \omega_{1,1} (f) d_x (z_i, z_{i+1}), \]

\[ d_y (w_i, w_{i+1}) \leq n^2 \omega (f, \delta_1, \delta_2) \]

\[ \omega_{1,1} (f, \delta_1, \lambda_1, \lambda_2, \delta_2) \leq n^2 \omega_{1,1} (f, \delta_1, \delta_2) = c \lambda_1 \lambda_2 \omega_{1,1} (f, \delta_1, \delta_2), \]

where \( c \) is an absolute constant that may differ from each step to another.
\[ F(x, y) = \left( \sum_{q=1}^{\ell} \sum_{i=1}^{r} \binom{r}{i} (-1)^{r-i-1} f \left( x - \frac{r h_2}{2} + i h_2, y_q \right) - \right. \\
\left. \sum_{j=1}^{k} \sum_{i=1}^{r} \binom{r}{i} (-1)^{r-i} f \left( x_j, y - \frac{r h_1}{2} + i h_1 \right) \right) \varphi_q(x), \psi_j(y) \]

belongs to \( \mathcal{L}_p(X) \otimes \psi_k + \Phi_1 \otimes \mathcal{L}_p(Y) \)

Therefore

\[ \|f - F\|_p \leq c(p) \sum_{q=1}^{\ell} \sum_{i=1}^{r} \| (-1)^r f \left( x - \frac{r h_2}{2}, y_q \right) + (-1)^r f \left( x - \frac{r h_2}{2}, y_q \right) \|
\]

\[ + \sum_{i=1}^{r} \binom{r}{i} (-1)^{r-i} f \left( x - \frac{r h_2}{2} + i h_2, y_q \right) \]

\[ + \sum_{j=1}^{k} \sum_{i=1}^{r} \binom{r}{i} (-1)^{r-i} f \left( x_j, y - \frac{r h_1}{2} + i h_1 \right) \varphi_q(x), \psi_j(y) \]

\[ \leq c(p) \sum_{q=1}^{\ell} \sum_{j=1}^{k} \omega_{r,r}(f, \delta_1, \delta_2)_p \varphi_q(X) \psi_j(y) \]

\[ = c(p) \omega_{r,r}(f, \delta_1, \delta_2)_p. \]

This completes the proof of 2.1.4.

**Proof of 2.1.5** Using definition of \( \omega_{r,r}(f, \delta_1, \lambda_1)_p \), we get

\[ \omega_{r,r}(f, \delta_1 + \delta_2, \lambda_1 + \lambda_2)_p = \sup_{0 < h_1 \leq \delta_1 + \delta_2} \sup_{0 < h_2 \leq \lambda_1 + \lambda_2} \| \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \left( f \left( X, y - \frac{r h_1}{2} + i h_1 \right) + f \left( X - \frac{r h_2}{2}, i h_2, y \right) \right) \|_p \]

\[ \leq c(p) \left( \omega_{r,r}(f, \delta_1, \lambda_1)_p + \omega_{r,r}(f, \delta_2, \lambda_2)_p + \omega_{r,r}(f, \delta_1, \lambda_2)_p + \omega_{r,r}(f, \delta_2, \lambda_1)_p \right) \]

Using (3) we obtain

\[ \omega_{r,r}(f, \delta_1 + \delta_2, \lambda_1 + \lambda_2)_p \leq c(p) \omega_{r,r}(f, \delta_1, \lambda_1)_p + \omega_{r,r}(f, \delta_2, \lambda_2)_p. \]

**Theorem 2.2.** For any two positive numbers \( \delta_1, \delta_2 \) and any \( f \in \mathcal{L}_p(X \times Y), p < 1 \) and \( X \) and \( Y \) are two compact metric space we have

\[ \inf_{\delta_1 > \delta_1} \inf_{\delta_2 > \delta_2} \omega_{r,r}(f, \delta_1, \delta_2)_p = \omega_{r,r}(f, \lambda_1, \lambda_2)_p \]

**Proof:** We must show that, if \( \delta_{1,n}, \delta_{2,n} \) are two decreasing sequences with limits \( \delta_1 \) and \( \delta_2 \) respectively we have

\[ \omega_{r,r}(f, \delta_{1,n}, \delta_{2,n})_p \text{ converges to } \omega_{r,r}(f, \delta_1, \delta_2) \text{ as } n \to \infty \]

Suppose there exists an \( \epsilon > 0 \) such that

\[ \omega_{r,r}(f, \delta_{1,n}, \delta_{2,n})_p > \omega_{r,r}(f, \delta_1, \delta_2) + \epsilon \]

This implies that there exist

\[ y = \frac{r h_{1,n}}{2} + i h_{1,n} \text{ in } Y, \text{ with } \]

\[ d_Y(y - \frac{r h_{1,n}}{2} + i h_{1,n}, y - \frac{r h_{1,n}}{2} + j h_{1,n}) < \delta_{1,n} \]

and

\[ x = \frac{r h_{2,n}}{2} + i h_{2,n} \text{ in } X, \text{ with } \]
\[ d_k \left( X - \frac{r h_2 n}{2} + i h_2 n, X - \frac{r h_2 n}{2} + j h_2 n \right) < \delta_{2,n} \]

Therefore
\[ \left\| \sum_{i=0}^{r} \left( \begin{array}{c} r \\ i \end{array} \right) (-1)^{r-i} \left( f (x, y \frac{r h_{1,n}}{2} + i h_{1,n}) + f (x - \frac{r h_{2,n}}{2} + i h_{2,n}, y) \right) \right\|_p > \omega_{r,r} (f, \delta_1, \delta_2)_p + \varepsilon \] (1)

Since \( X \) and \( Y \) are compact spaces, we get the above two sequences in \( X \) and \( Y \) have converging subsequences. This leads to \( h_{1,n_k} \to h_{1,\alpha} \) and \( h_{2,n_k} \to h_{2,\alpha} \), \( h_{1,\alpha}, h_{2,\alpha} \in X \) and \( h_{2,\alpha} \in Y \) so
\[ \left\| \sum_{i=0}^{r} \left( \begin{array}{c} r \\ i \end{array} \right) (-1)^{r-i} \left( f (x, y \frac{r h_{1,n_k}}{2} + i h_{1,n_k}) + f (x - \frac{r h_{2,n_k}}{2} + i h_{2,n_k}, y) \right) \right\|_p \]

Converges to
\[ \left\| \sum_{i=0}^{r} \left( \begin{array}{c} r \\ i \end{array} \right) (-1)^{r-i} \left( f (x, y - \frac{r h_{1,0}}{2} + i h_{1,0}) + f (x - \frac{r h_{2,0}}{2} + i h_{2,0}, y) \right) \right\|_p \]

From (1) we have:
\[ \left\| \sum_{i=0}^{r} \left( \begin{array}{c} r \\ i \end{array} \right) (-1)^{r-i} \left( f (x, y - \frac{r h_{1,0}}{2} + i h_{1,0}) + f (x - \frac{r h_{2,0}}{2} + i h_{2,0}, y) \right) \right\|_p \geq \omega_{r,r} (f, \delta_1, \delta_2)_p + \varepsilon. \]

But \( h_{1,n_k} \to h_1 \) and \( h_{1,n_k} < \delta_{1,n_k} \to \delta_1 \).

Also
\[ h_{2,n_k} \to h_2 \] and \( h_{2,n_k} < \delta_{2,n_k} \to \delta_2 \). Therefore
\[ \omega_{r,r} (f, \delta_1, \delta_2) \geq \left\| \sum_{i=0}^{r} \left( \begin{array}{c} r \\ i \end{array} \right) (-1)^{r-i} \left( f (x, y - \frac{r h_{1,0}}{2} + i h_{1,0}) + f (x - \frac{r h_{2,0}}{2} + i h_{2,0}, y) \right) \right\|_p \]
\[ \geq \omega_{r,r} (f, \delta_1, \delta_2)_p + \varepsilon. \]

Which is a contradiction. Thus our result is satisfied.

We can strengthen the above result by the following example it mean the above result not true in general.

**Example 2.3.** Define \( G: \{0, 1\} \times \{0, 1\} \to R, \) as
\[ G (z, w) = \begin{cases} 1 & (z, w) = (1, 1) \\ 0 & (z, w) \neq (1, 1) \end{cases} \]

It is clear that \( G \) is a continuous function
\[ \omega_{r,r} (f, \delta_1, \delta_2)_p = \begin{cases} 1 & \delta_1, \delta_2 \geq 1 \\ 0 & \text{otherwise} \end{cases} \]

If we choose \( \delta_1, \delta_2 = 1 \), we get that the above result is not true.

**3. A version of Jackson Theorem**

A classical type theorem due to Jackson, for the approximation of functions \( f \in L^p[a, b] \) by polynomials says
\[ E_n (f)_p \leq c(p) \omega_r (f, \frac{1}{n})_p, \quad \infty \geq p > 0, \] (2)
where \( c(p) \) is a positive constant depends on \( p \) only, for \( p < 1, \)
\[ E_n (f)_p = \inf_{f \in P_n} \| f - p \|, \]
\( P_n \) is the space of polynomials of degree less than or equal to \( n \), and
\[ \omega_r (f, \delta) = \sup_{|h| \leq \delta} \| \Delta_h^n f \|_p \]

The inequality (2) was given in terms of the nth entropy number
\[ \delta_n ([a, b]) = \frac{b-a}{2^n}, \]
which generalized using the compact metric space \( X \).

In [Stephani, 1992] I. Stephani was proved
\[ E_n (f) = \omega_1 (f, \in_n (X)), \]
where \( \omega_1 \) is the modulus of continuity of one variable for a function \( f \in C (X) \), and \( E_n (f) \) is the error of the function \( f \) to some class \( \Phi_n \) in
\[ \emptyset_1 \subseteq \emptyset_2 \subseteq \ldots \subseteq \emptyset_n \subseteq \ldots \]
with union dense in \( C (X) \).

Now let us introduce the blending Jackson version theorem
\[ E_{m,n} (f) \leq c(p) \omega_{t,s} (f, \frac{1}{m}, \frac{1}{n})_p, \]
where \( f \) defined on \( X \times Y \), and,
\[ E_{m,n} (f) = \inf \| f - P_{m,n} \|_{p(X \times Y)}, \]
the infimum is taken on all pseudo polynomials that have the form
\[ P_{m,n} (x,y) = \sum_{i=0}^n \alpha_i (x)y^i + \sum_{j=0}^m \beta_j (y) x^j \]

And \( \alpha_i \) and \( \beta_j \) are bounded function coefficients. Inequality (3) was proved by Yu. A. Brudnyi [Brudnyi, 1992; Gonska, Jetter, 1985]. By [Hbing,1949] for \( \omega_{1,1} \), and a continuous function \( f \), \( X = [a, b], y = [c,d] \). And (3) also proved in [Gonska, 1985; Jetter,1989; Cottin,1988; Cottin,1992] for blending Jackson theorem using trigonometric pseudo polynomials and continuous function in \( C(X) \).

Let us define the blending space \( C(X) \oplus M(Y) + M (X) \oplus C(Y) = B \mathcal{L} \), with respect to a suitable norm \( M(Y) \) and \( M (X) \) space of bounded functions equipped with the uniform norm on the compact metric space \( X \) or \( Y \). \( \oplus \) is the tensor product defined by \( f_1 \otimes f_2 \in C(X) \otimes M (Y) \), defined by
\[ f_1 \otimes f_2 (x,y) = f(x)g(y) \]

Let \( X \) be a compact metric space under the metric \( d_x \), with
\( \psi_1 \subseteq \psi_2 \subseteq \ldots \subseteq \psi_n \subseteq \ldots \), its nested subspaces and partition.
\[ E_{m,n}(f) = \inf\{ \| f - P_{m,n} \| : \varphi_m, \psi_m \}, \]
in\( f \) is on all pseudo polynomials:
\[ P_{m,n} (x,y) = \sum_{i=0}^m A_i (y) x^i + \sum_{j=0}^n B_j (x) y^j, \]
where \( A_i, B_j \) are bounded functions, is the degree of the approximation of \( f \) using the blending space of pseudo polynomials as an approximation space.
\[ B( M(X), M(Y), A_x, A_y) = A_x \otimes M(y) + M(X) \otimes A_y \]
If \( X \) is a compact space under the metric \( d_x \), a partition of unity \( \varphi_1, \varphi_2, \ldots, \varphi_n \) on \( X \), it mean \( \varphi_j \in C(X) \),
\[ 0 \leq \varphi_j (t) \leq t, \quad \sum_{j=1}^n \varphi_j (t) = 1, \quad t \in X, \quad j \text{ is natural} \]
with \( n \) greater than 2, to be controllable if the supports
\[ \text{supp} ( \varphi_j ) = \{ t \in X : \varphi_j (t) \neq 0 \} \]
Have the property
\[ \epsilon_1 (\text{supp} ( \varphi_j )) < \epsilon_{n-1} (X), j = 1, 2, \ldots, n. \]
**Theorem 3.1:** If $(X, \| \cdot \|_p)$ and $(Y, \| \cdot \|_p)$ are compact quasi normed spaces for $0 < p < 1$, and let $f \in L^p(X \times Y)$, then

$$E_{m,n}(f)_p \leq \inf_{\delta_1 > \delta_{1,m}(X)} \omega_{r,r} (f, \delta_1, \delta_2)_p$$

(4)

**Proof:** Let $m = 1$, assume $\delta_1 > \delta_{1,1}(X)$, then we can find $x_1 \in X$, satisfy $X \subseteq B(x_1, \delta_1)$. Also if $n = 1$, assume $\delta_2 > \delta_{2,1}(Y)$ then we can fined $y_1 \in Y$, satisfy $Y \subseteq B(y_1, \delta_2)$. Then we map

$$F(x, y) = \sum_{i=1}^{r} \binom{r}{i} (-1)^{r-i-1} \left( f(x, y - \frac{r h_1}{2} + i h_1) + f(x - \frac{r h_2}{2} + i h_2, y) \right)$$

$\in L^p(X) \otimes \psi_1 + \Phi_2 \otimes L^p(Y)$. Therefore we have

$$|f(x, y) - F(x, y)| \leq C \left| (-1)^{r} f(x, y - \frac{r h_1}{2}) + f(x - \frac{r h_2}{2}, y) - \sum_{i=1}^{r} \binom{r}{i} (-1)^{r-i-1} \left( f(x, y - \frac{r h_1}{2} + i h_1) + f(x - \frac{r h_2}{2} + i h_2, y) \right) \right|.$$ 

Therefore

$$\|f - F\|_{L^p(X \times Y)} \leq c(p) \omega_{r,r} (f, \delta_1, \delta_2)_p.$$ 

Thus the inequality of our theorem is satisfied for $m = n = 1$, for any $\delta_1 > \delta_{1,1}(X)$ and $\delta_2 > \delta_{2,1}(Y)$. If $m = 1$ and $n > 1$, we can find $k \leq n$, satisfy $\delta_{1,n}(Y) = \delta_{1,k}(Y)$. If $k = 1$, we can choose $\delta_2 > \delta_{1,1}(Y)$, and apply the same lines of the case above. Then $k > 1$, implies $\delta_{1,k}(Y) < \delta_{1,k-1}(Y)$, so we have $\delta_{1,k}(Y) < \delta < \delta_{1,k-1}(Y)$. Using the entropy definition we can fined the points $x_1, x_2, \ldots, x_k \in Y$ such that

$$Y \subseteq \bigcup_{j=1}^{k} B_{\delta_2}(x_j)$$

Using the same lines used in [Cottin, 1988] we can get a partition $\psi_1, \psi_2, \ldots, \psi_k$, satisfying

$$\text{supp} (\psi_j) \subseteq B_{\delta_2}(x_j), j = 1, 2, \ldots, k.$$ 

Then since $\delta_2 < \delta_{1,k-1}(Y)$, we can obtain $(\psi_j)_{j=1}^{k}$ satisfy the condition of controllability. The map

$$F(X, y) = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i-1} f \left( x - \frac{r h_2}{2} + i h_2, y \right) - \sum_{j=1}^{k} \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} f \left( x - \frac{r h_2}{2} + i h_2, y_j \right) \psi_j (y),$$

belongs to $L^p(X) \otimes \psi_k + \Phi_2 \otimes L^p(Y)$, and $BL (L^p(X), L^p(Y), \Phi_1, \psi_n)$. We have using the conditions of controllability that

$$\|f - F\|_p \leq c(p) \sum_{j=1}^{r} \|(-1)^{r} f \left( x - \frac{r h_2}{2}, y \right) + \sum_{i=1}^{r} \binom{r}{i} (-1)^{r-i} f \left( x - \frac{r h_2}{2}, y \right) \psi_j (y)|$$

$$\psi_j (y) \leq c(p) \sum_{j=1}^{k} \omega_{r,r} (f, \delta_1, \delta_2)_p \psi_j (y) = c(p) \omega_{r,r} (f, \delta_1, \delta_2)_p$$

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Thus (4) satisfied when \( m = 1 \), and \( n > 1 \). Using the same lines above we can prove the case when \( m > 1 \) and \( n = 1 \). It remain the case when \( m, n > 1 \). Let \( \ell \) and \( k \) are two naturals with \( \ell \leq m, k \leq n \) and \( \delta_{1,\ell}(X) = \delta_{1,m}(X) \) and \( \delta_{1,k}(Y) = \delta_{1,n}(Y) \). When \( \ell = k = 1 \), we shall return to the case above. Let us assume \( k, \ell > 1 \), we shall prove (4) for \( \delta_1 \) and \( \delta_2 \), satisfying \( \delta_{1,\ell}(X) < \delta_1 < \delta_{1,\ell - 1}(X) \) and \( \delta_{2,k}(Y) < \delta_2 < \delta_{2,k-1}(Y) \). By entropy numbers definition, we can find \( x_1, x_2, \ldots, x_\ell \) and \( y_1, y_2, \ldots, y_k \in Y \), satisfying

\[
X \subseteq \bigcup_{q=1}^{\ell} B_{x_q}^{(1)}(\delta_1), \quad Y \subseteq \bigcup_{j=1}^{k} B_{y_j}^{(2)}(\delta_2)
\]

As in the case above, the partition of unity \( (\varphi_q), (\psi_j) \) subordinate to the open cover in (5) satisfying the condition of controllability because of \( \delta_1 < \delta_{1,\ell-1}(X), \delta_2 < \delta_{2,k-1}(Y) \). Then define the

**Theorem 3.4.** Let \( X \) have the Peano property, and let \( P \) be a positive linear operator from \( L^p(X) \) to \( L^p(X) \), satisfying \( P(f(x)) = \bar{f}(x) \), where \( f(x) \) is the identity function. Then for any \( f \in L^p(X) \), and \( \delta > 0 \) we have \( \|P(f) - f\|_p \leq c(p) \omega_{r,r}(f, \delta_1, \delta_2)_p \)

**Proof:** Since \( G \) satisfy Peano property, so for any two points with distance \( \leq \delta_1 \) or \( \leq \delta_2 \), we have

\[
\left\| \sum_{i=0}^{r} R(-1)^{r-i} \left( f(x, y - \frac{rh_1}{2}, ih_1) + f \left( x - \frac{rh_2}{2}, ih_2, y \right) \right) \right\|
\leq \omega_{r,r}(f, \delta_1, \delta_2)_p \sum_{i=0}^{r} 1 + \frac{d ((x, y - \frac{rh_1}{2} + ih), (x - \frac{rh_2}{2} + ih_2, y))}{\min{\delta_1, \delta_2}}
\]

Then

\[
\|P(f) - f\|_p \leq c(p) \omega_{r,r}(f, \delta_1, \delta_2)_p .
\]

**References**


