

Relationship Between Moduli of Smoothness of Function in Two Different Spaces

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Abstract

In this paper we prove an improved version of P.L .UL'yanov inequality in terms of the fractional order modulus of smoothness of 2π periodic function in different spaces.

Keywords: Direct Theorem, fractional difference, L_p spaces.

الخلاصة

برهنا في هذا البحث احد أنواع متراجحة Ulyano'v بدلالة مقياس النعومة ذي الرتبة الكسرية للدوال دورة 2π في فضاءين مختلفين .
الكلمات المفتاحية: المبرهنة المباشرة، الفرق ذي الرتبة الكسرية، الفضاء L_p .

1- Introduction

In [UL'yanov1968, UL'yanov 1970] UL'yanov proved if $g(x) \in L_p, 1 \leq p < q < \infty$, for any $k \in (0,1]$ then

$$\omega_1(g, k)_q \leq c_1 \left\{ \int_0^k \left(s^{-\left(\frac{1}{p}-\frac{1}{q}\right)} \omega_1(g, s)_p \right)^q \frac{ds}{s} \right\}^{1/q} \quad (1-1)$$

where c_1 is a positive constant does not depend on g and k .

In [Kolyada 1988] Kolyada strength (1-1) and proved if $g(x) \in L_p, 1 < p < q < \infty$, then for any $k \in (0,1]$ we have

$$k^{1-\frac{1}{p}+\frac{1}{q}} \left\{ \int_k^1 \left(t^{\frac{1}{p}-\frac{1}{q}-1} \omega_1(g, t)_q \right)^p \frac{dt}{t} \right\}^{1/p} \leq c_2 \left\{ \int_0^k \left(t^{-\left(\frac{1}{p}-\frac{1}{q}\right)} \omega_1(g, t)_p \right)^q \frac{dt}{t} \right\}^{1/q} ,$$

where c_2 is a positive constant does not depend on g and k .

In this article we improve the above inequalities in terms of module of higher orders i.e. we prove these inequalities for $0 < p, q < 1$.Now let us introduce some definitions that can be used in this paper.

Let $L_p[0,2\pi], 0 < p < 1$, be the space of periodic measurable function $g(x)$ with period 2π , under the norm $\|g\|_p = \left(\int_0^{2\pi} |g(x)|^p dx \right)^{\frac{1}{p}} < \infty$. The β th - difference of $g(x)$ at the point $x, x \in R$ of fractional order β ($\beta > 0$) is given by

$$\Delta_t^\beta g(x) = \sum_{i=0}^{\infty} (-1)^i \binom{\beta}{i} f(x + (\beta - i)t) \text{ [Tikhonov, 2004]}$$

The β th modulus of smoothness in terms of fractional order ($\beta > 0$) is defined by

$$\omega_\beta(g, t)_p = \sup_{|h| \leq t} \|\Delta_h^\beta g(x)\| \text{ [Butzer, Dyckhoff and Goerlich 1977]}$$

The K - functional is defined by

$$K(g, t^\beta, L_p, w_p^\beta) = \inf_{f \in w_p^\beta} \left(\|g - f\|_p + \|D_f^\beta\|_p \right) \text{ [Tikhonov 2004]}$$

And $(a_n \downarrow 0), (n \uparrow \infty)$ means a_n is decreasing sequence converge to 0 as n approach to ∞ .

2-The Auxiliary Results

In this section we introduce some results that we need to prove of our Theorem

Lemma 2.1 [Sharba 2012]

$w_\beta(g, \lambda)_p$ is non decreasing non negative function of λ on $(0, \infty)$ with $\lim_{\lambda \rightarrow 0^+} \omega_\beta(g, \lambda)_p = 0$

Lemma 2.2 [Sharba 2012]

For any $g \in L_p[-1, 1], 0 < p < 1$, we have

$$E_n(g)_p \leq C(p) \omega_r(g, n^{-1})_p n > r$$

Lemma 2.3 [Sharba 2012]

For a function $f \in L_p[a, b], 0 < p < 1$ we have

$$\omega_\beta(f, t)_p \approx K(f, t^\beta, L_p, w_p^\beta)$$

Lemma 2.4 [Nikol'skii 1975]

Let $a_n \geq 0, 0 < \alpha \leq \beta < \infty$. Then

$$\left(\sum_{n=1}^{\infty} a_n^\beta \right)^{\frac{1}{\beta}} \leq \left(\sum_{n=1}^{\infty} a_n^\alpha \right)^{\frac{1}{\alpha}}$$

Lemma 2.5

Let $g(x) \in L_p[0, 2\pi], 0 < p < 1$ we have

$$E_{2^n}(g)_q \leq c \left(\sum_{m=n}^{\infty} 2^{mq \left(\frac{1}{p} - \frac{1}{q} \right)} E_{2^m}^q(g)_p \right)^{\frac{1}{q}},$$

where c is positive constant does not depend on g and n .

Proof:

Using the same way used in Theorem (1) of [Potapov, Simonov and Tikhonov 2008]

Proposition 2.6

Let $f(x) \in L_p[0, 2\pi]$, $0 < p < 1$, $\beta > 0$. Then for any $m \in N$

i) $E_m(f)_p \leq c_3 \omega_\beta \left(f, \frac{1}{m} \right)_p,$

ii) $c_4(m^{-\beta} \|S_m^{(\beta)}(f, x)\|_p + E_m(f)_p) \leq \omega_\beta \left(f, \frac{1}{m} \right)_p \leq c_5(m^{-\beta} \|S_m^{(\beta)}(f, x)\|_p + E_m(f)_p),$

iii) $c_6 \left\| \sum_{n=1}^m n^\beta A_n(x) \right\|_p \leq \|S_m^{(\beta)}(f, x)\|_p \leq c_7 \left\| \sum_{n=1}^m n^\beta A_n(x) \right\|_p,$

where c_3, c_4, c_5, c_6, c_7 are positive constants do not depend on f and m .

Proof (i): Follows directly from Lemma 2.2

Proof (ii): Follows directly from Lemma 2.3 and part (i)

Proof (iii): Follows directly using the hypothesis

$$S_m^{(\beta)}(x) \sim \sum_{n=0}^{\infty} (n+1)^{-\beta} \epsilon_n A_n(x) \text{ such that } |\epsilon_n| \leq 1$$

$$c_8 \left\| \sum_{n=1}^m (n+1)^{-\beta} \epsilon_n A_n(x) \right\|_p \leq \|S_m^{(\beta)}(f, x)\|_p \leq c_9 \left\| \sum_{n=1}^m (n+1)^{-\beta} \epsilon_n A_n(x) \right\|_p$$

□

Proposition 2.7

Let $f(x) \in L_p[0, 2\pi]$ $0 < p < 1$, $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$, where $a_n \downarrow 0$ ($n \uparrow \infty$). Then

$$c_{10} \left(a_0^p + \sum_{n=1}^{\infty} a_n^p n^{p-2} \right)^{\frac{1}{p}} \leq \|f\|_p \leq c_{11} \left(a_0^p + \sum_{n=1}^{\infty} a_n^p n^{p-2} \right)^{\frac{1}{p}},$$

where the positive constants c_{10}, c_{11} do not depend on f .

Proof:

$$\begin{aligned} \|f\|_p &= \left\| \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \right\|_p \\ &= \left(\int_0^{2\pi} \left| \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \right|^p dx \right)^{\frac{1}{p}} \\ &\leq \left(\int_0^{2\pi} \left| \frac{a_0}{2} \right|^p + \sum_{n=1}^{\infty} |a_n|^p |\cos nx|^p dx \right)^{\frac{1}{p}} \\ &\leq c(p) \left(|a_0|^p + \sum_{n=1}^{\infty} |a_n|^p n^p \right)^{\frac{1}{p}} \\ \|f\|_p &= \left(\int_0^{2\pi} \left| \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \right|^p dx \right)^{\frac{1}{p}} \end{aligned}$$

Since $\cos nx$ is bounded on $[0, 2\pi]$ and $a_n \rightarrow 0$ so it is also bounded and the sum of any two bounded function it is also bounded functions so that

$$\|f\|_p \geq C(p) \left(|a_0|^p + \sum_{n=1}^{\infty} |a_n|^p n^{p-2} \right)^{\frac{1}{p}} \quad \square$$

3- The main result

In this section we introduce our main results.

Theorem 3.1

Let $g(x) \in L_p, 0 < p < q < 1, \beta > 0, g(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, a_n n^{\alpha + \frac{1}{p} - \frac{1}{q}} \downarrow 0, (n \uparrow \infty)$. Then for any $k \in (0, 1]$ we have :

$$\omega_{\beta}(g, k)_q \leq c_{12} \left\{ \left(\int_0^k t^{-\frac{1}{p} + \frac{1}{q}} \omega_{\beta + \frac{1}{p} - \frac{1}{q}}(g, t)_p \right)^q \frac{dt}{t} \right\}^{\frac{1}{q}},$$

where c_{12} is a positive constant does not depend on g and k .

Proof:

Using Lemma 2-1 and applying Proposition 2.6 for any $n = 0, 1, 2, \dots$ we get

$$\omega_{\beta}\left(g, \frac{1}{2^n}\right)_q \leq c(q) \omega_{\beta}\left(g, \frac{1}{2^{n+1}}\right)_q \leq c(q) \left(2^{-(n+1)\beta} \left\| \sum_{n=1}^{2^{n+1}} (n+1)^{-\beta} \epsilon_n A_n(x) \right\|_q + E_{2^{n+1}}(g)_q \right)$$

Using Proposition 2.7 and Lemma 2.4 we get

$$\begin{aligned} \left\| \sum_{n=1}^{2^{n+1}} (n+1)^{-\beta} \epsilon_n A_n(x) \right\|_q &\leq c(q) \left(\sum_{n=1}^{2^{n+1}} \epsilon_n^q (n+1)^{-\beta q} (n+1)^{q-2} \right)^{\frac{1}{q}} \\ &\leq c(q) \left(\sum_{n=1}^{2^n} \epsilon_{2^n}^q 2^{-(n+1)\beta q} 2^{(n+1)q(1-\frac{1}{q})} \right)^{\frac{1}{q}} \\ &\leq c(p) \left(\sum_{n=1}^{2^n} \epsilon_{2^n}^p 2^{-(n+1)\beta p} 2^{(n+1)p(1-\frac{1}{q})} \right)^{\frac{1}{p}} \\ &\leq c(p) \left(\sum_{n=1}^{2^n} \epsilon_n^p (n+1)^{p(\beta + \frac{1}{p} - \frac{1}{q})} (n+1)^{p-2} \right)^{\frac{1}{p}} \\ &\leq c(p) \left\| \sum_{n=1}^{2^n} (n+1)^{\beta + \frac{1}{p} - \frac{1}{q}} \epsilon_n A_n(x) \right\|_p \end{aligned} \quad \left. \vphantom{\sum_{n=1}^{2^n}} \right\} (3-1)$$

Combining estimates in Lemma (2.5) and (3-1) and applying Proposition 2.6 we obtain

$$\omega_{\beta}\left(g, \frac{1}{2^n}\right)_q \leq c(p) \left(\sum_{n=0}^{2^n} 2^{(n+1)q\left(\frac{1}{p}-\frac{1}{q}\right)} \omega_{\beta+\frac{1}{p}-\frac{1}{q}}^q\left(g, \frac{1}{2^{n+1}}\right)_p \right)^{\frac{1}{q}},$$

Using properties of integral we finally get:

$$\omega_{\beta}(g, k)_q \leq c(p) \left\{ \int_0^k \left(t^{-\left(\frac{1}{p}-\frac{1}{q}\right)} \omega_{\beta+\frac{1}{p}-\frac{1}{q}}^q(g, t)_p \right)^q \frac{dt}{t} \right\}^{\frac{1}{q}},$$

Thus the proof is complete.

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