Fully Dual-Stable S-system

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Abstract

An S-system M is fully stable if $\alpha(N) \subseteq N$ for each subsystem N and S–homomorphism $\alpha$ of N into M. In this paper we study the dual concept of full stability. Duo property of an S-system being a necessary condition for both full stability and full dual stability, and quasi-projectivity is sufficient condition for duo to be fully dual stable system. Several properties and characterizations of full dual stability are investigated.

Keywords: dual-stable subsystem, fully dual-stable S-system, duo S-system, quasi projective S-system, Hopfian and Co-Hopfian S-system.

1. Introduction and Preliminaries

The notion of full stability and full dual stability were studied on modules by (Abbas, 1990; Abbas and Al-Hosainy, 2012). Most of the modules notions, were reversed to S-system (S-acts), and interesting results were obtained. The notions of full stability on S-system, were studied, recently, by Abbas and Baanno (Abbas and Baanoon, 2015). In this paper, the notion of full dual-stability on S-system, is investigated.

A subsystem $B_S$ of an S-system $A_S$ is said to be dual stable if $B_S \times B_S \subseteq \ker \alpha$, for each S-homomorphism $\alpha : A_S \rightarrow A_S/B_S$. The S-system $A_S$ is said to be fully dual stable (shortly, fully d-stable) if each subsystem of $A_S$ is dual-stable. An S-system $A_S$ is said to be strongly dual stable if $\ker g \subseteq \ker f$ whenever $f$ and $g$ are S-homomorphism of $A_S$ into $B_S$ and $g$ is surjective, where $B_S$ is any S-system.

The above two conditions are equivalent in the case of modules, but in S-system, the second condition implies the first. This difference occurred because of the fact that in modules, the kernel of a homomorphism is a submodule, while the kernel of S-homomorphism does not need to be induced by a subsystem.

In this section, some preliminaries about S-system and related concepts were given. For more information about S-system (s-act) see (Kilp and Mikhlev, 2000).

In section 2, the main results about full d-stability and the related concepts(duo, multiplication, quasi-projective) were given. More results about full d-stability and quasi-projectivity discussed in section 3.
(1.1) Definition (Kilp and Mikhalev, 2000). Let $S$ be a monoid and $A$ is a nonempty set. If we have a mapping $\mu : A \times S \rightarrow A$, $(a, s) \mapsto a \cdot s = \mu(a, s)$ such that:

a) $a \cdot 1 = a$ and
b) $a(st) = (as)t$ for $a \in A$, $s, t \in S$,

we call $A$ a right $S$–system or right system over $S$ and write it as $A_S$. More informally, we often say that $\mu$ defines a right multiplication of element from $A$ by element of $S$.

Analogously, we define a left $S$-system $A$ and write $S_A$.

(1.2) Definition (Abbas and Dahash, 2014). A subsystem $N$ of $S$-system $M$ is called fully invariant if $\alpha(N) \subseteq N$ for each $S$-endomorphism $\alpha$ of $M_S$, $M_S$ is duo $S$-system if each subsystem of $M_S$ is fully invariant.

The following lemma is a part of lemma (1.3) (Roueentan and Ershad, 2012), for completeness we give a proof for it.

(1.3) Lemma: An $S$-system $M_S$ is duo iff for each endomorphism $f$ of $M$ and each element $m$, there exists $r \in S$ (depending on $m$) such that, $f(m) = mr$.

Proof: ($\Rightarrow$) Assume $M$ is a duo $S$-system, $f \in \text{End}_S M$ and $m \in M$. Then $mS$ is a subsystem of $M$ and $m \in mS$.

Since $M$ is duo, we have $f(mS) \subseteq mS$, hence $f(m) \in mS$, that is $\exists r \in S$ such that $f(m) = mr$.

($\Leftarrow$) It is clear.

Recall that $U$ is said to be a generating set of $A_S$ if for all $a \in A$, $a = us$ for some $u \in U$ and $s \in S$.

(1.4) Definition (Kilp and Mikhalev, 2000): A set $U$ of generating elements of a right $S$-system $A_S$ is said to be a basis of $A_S$ if every element $a \in A_S$ can be uniquely presented in the form $a = us$, $u \in U$, $s \in S$, if $a = u_1s_1 = u_2s_2$, then $u_1 = u_2$ and $s_1 = s_2$. If an $S$-system $A_S$ has a basis $U$, then it is called a free $S$-system or, more precisely, a $|U|$-free $S$-system. In particular, $S_S$ is 1-free with basis $\{1\}$. Also, we say that $A_S$ is of rank $|U|$.

(1.5) Definition (Kilp and Mikhalev, 2000): Let $M_S$ be an $S$-system. An equivalence relation $\rho$ on $M$ is called an $S$-system congruence or a congruence on $M_S$, if $(m, n) \in \rho$ implies $(ms, ns) \in \rho$ for $m, n \in M_S$, $s \in S$. If $S$ is a monoid then any right(semigroup) congruence $\rho$ on $S$ is an $S$-system congruence on $S_S$.

(1.6) Definition (Kilp and Mikhalev, 2000): Any subsystem $B_S \subseteq A_S$ defines the Rees congruence $\rho_B$ on $A$, by setting $(a, \dot{a}) \in \rho_B$ if $a, \dot{a} \in B$ or $a = \dot{a}$.

(1.7) Definition (Kilp and Mikhalev, 2000): Let $A_S$ be a right $S$-system. An element $\theta \in A_S$ is called a fixed element of $A_S$ if $\theta s = \theta$ for all $s \in S$. If $A_S$ has a unique fixed element $\theta$, then $\theta$ is called zero element of $A_S$, we will denote the zero element of $A_S$ by $0$.

(1.8) Definition (Kilp and Mikhalev, 2000): We call an $S$-system $A_S$ decomposable if there exist two subsystems $B_S, C_S \subseteq A_S$ such that $A_S = B_S \cup C_S$ and $B_S \cap C_S = \emptyset$. In this case $A_S = B_S \cup C_S$ is called a decomposition of $A_S$. Otherwise $A_S$ is called indecomposable.

If we consider $S$-system with zero $\theta$, then we have to replace $\emptyset$ by $\{\theta\}$ to define decomposable and indecomposable $S$-system with zero.
(1.9) **Definition** (Kilp and Mikhalev, 2000): An S-system $A_S$ is called **torsion free** if for any $x, y \in A_S$ and right cancellable element $c \in S$ the equality $xc = yc \Rightarrow x = y$.

(1.10) **Definition** (Kilp and Mikhalev, 2000): An S-system $A_S$ is called quasi-projective if for any epimorphism $\pi : A_S \rightarrow B_S$ and homomorphism $\alpha : A_S \rightarrow B_S$ there exists an endomorphism $f$ of $A_S$ such that $\pi f = \alpha$.

**Note** it is clear that if $M_S$ is quasi-projective then for all $N$ subsystem $M_S$ and for all $\alpha : M \rightarrow M / N$ there exists $f \in \operatorname{End} M$ such that $\pi_{N^o} f = \alpha$ where $\pi_N$ is the natural epimorphism of $M$ onto $M / N$.

(1.11) **Definition** (Kilp and Mikhalev, 2000): Let $A_S$ and $B_S$ be two S-systems. Consider an S-homomorphism $f : A_S \rightarrow B_S$. Then $f$ is called a **retraction** if $f$ is right invertible, i.e. there exists $g \in \operatorname{Hom}_S(B,A)$ with $fg = \text{id}_B$; $B$ is called a retract of $A$.

2. **Fully dual stable S-system**

We start by introducing the dual concept of fully stable S-system.

(2.1) **Definition**: Let $M$ be an S-system and $N$ is a subsystem of $M$. $N$ is said to be d-stable sub-system of $M$ if, $N \times N \subseteq \ker \alpha$, for all $\alpha : M \rightarrow M / N$. $M$ is said to be fully d-stable if, any subsystem of $M$ is d-stable.

(2.2) **Remark**: If $S$ is a monoid, then $S$ is fully d-stable if $S_S$ is fully d-stable.

(2.3) **Lemma**: If $f : A_S \rightarrow B_S$ is homomorphism, $\varphi : B_S \rightarrow C_S$ is an isomorphism then $\ker \varphi \circ f = \ker f$.

**Proof**: Is clear.

(2.4) **Proposition**: If $M_S$ is fully d-stable S-system, then $M / N$ is fully d-stable for any subsystem $N_S$ of $M_S$.

**Proof**: Let $K / N$ be a subsystem of $M / N$ and $\alpha : M / N \rightarrow (M / N) / (K/N)$ be a homomorphism, consider the composition

\[
M \xrightarrow{\pi_S} M/N \xrightarrow{\alpha} (M/N)/(K/N) \cong M/K
\]

since $M$ is fully d-stable, it follows, $K \times K \subseteq \ker \varphi \alpha \pi_N = \ker \alpha \pi_N$ (lemma 2.3). Now, if $([k_1]_N, [k_2]_N) \in K / N \times K / N$, then $(k_1, k_2) \in K \times K$, hence $(\alpha \pi_N)(k_1) = (\alpha \pi_N)(k_2)$, that is $\alpha(\pi_N(k_1)) = \alpha(\pi_N(k_2))$. Then $\alpha([k_1]_N) = \alpha([k_2]_N)$, that is $([k_1]_N, [k_2]_N) \in \ker \alpha$. Therefore $K / N \times K / N \subseteq \ker \alpha$, and $M / N$ is fully d-stable.

(2.5) **Theorem**: Let $M$ be an S-system. The following two statements are equivalent.

1. For each congruence $\rho$ on $M$ and for each S-homomorphism $\alpha : M \rightarrow M / \rho$, $\rho \subseteq \ker \alpha$ holds.
2. For any S-system $A_S$, and for each two homomorphisms $f, g : M_S \rightarrow A_S$, with $g$ onto, $\ker g \subseteq \ker f$ holds.

![Diagram](image-url)
Proof:- (1) $\Rightarrow$ (2)  
Since $g: M_S \rightarrow A_S$ is onto hence $A_S \cong M_S / \ker g$, let $\varphi: A_S \rightarrow M_S / \ker g$ be an isomorphism therefore $\varphi \circ f: M_S \rightarrow M_S / \ker g$. By hypothesis, $\ker g \subseteq \ker (\varphi \circ f) = \ker f$ (since $\varphi$ is one to one).

(2) $\Rightarrow$ (1)  
$\rho$ congruence on $M_S$, $\alpha: M_S \rightarrow M_S / \rho$, let $\pi: M_S \rightarrow M_S / \rho$ natural epimorphism $\pi(m) = [m]_\rho$, then $\ker \pi = \rho$, $([m_1]_\rho = [m_2]_\rho \Leftrightarrow (m_1, m_2) \in \rho)$. By (2) $\ker \pi \subseteq \ker \alpha$, therefore $\rho \subseteq \ker \alpha$.

(2.6) Definition: An $S$-system $M_S$ is said to be strongly d-stable $S$-system if it satisfies any of the two equivalent conditions of Theorem (2.5).

(2.7) Remark: Any strongly d-stable $S$-system is fully d-stable.
Proof: Let $N_S$ be a subsystem of $M_S$ and $\alpha: M_S \rightarrow M_S / N_S$, then $\rho_N$ is a congruence on $M$ and $N \times N \subseteq \rho_N$ (where $\rho_N$ is the Rees congruence on $M$). Since $M_S$ is strongly fully d-stable then $\rho_N \subseteq \ker \alpha$, and hence $N \times N \subseteq \ker \alpha$.  

(2.8) Proposition: A homomorphic image of a strongly d-stable $S$-system is strongly d-stable $S$-system too.
Proof: Assume that $f: M \rightarrow \hat{M}$ is an epimorphism. Let $g, h: \hat{M} \rightarrow A$ be two $S$-homomorphisms with $g$ surjective, then $g \circ f, h \circ f: M \rightarrow A$ are $S$-homomorphisms, we have $\ker g \circ f \subseteq \ker h \circ f$, (M is strongly d-stable, $g \circ f$ onto).

\[
\begin{array}{ccc}
M & \xrightarrow{f} & \hat{M} \\
& \downarrow & \downarrow h \\
M & \xrightarrow{g} & A
\end{array}
\]

To prove $\ker g \subseteq \ker h$, let $(m, n) \in \ker g$, then $g(m) = g(n)$ since $f$ is surjective we have $m = f(x), n = f(y)$ for some $x, y \in M$, hence $g(f(x)) = g(f(y))$, that is $(x, y) \in \ker g \circ f \subseteq \ker h \circ f$, hence $(x, y) \in \ker (h \circ f) \Rightarrow h(f(x)) = h(f(y))$, $h(m) = h(n) \Rightarrow (m, n) \in \ker h$. Therefore, $\ker g \subseteq \ker h$, which implies $\hat{M}$ is strongly fully d-stable.

(2.9) Proposition: Let $M$ be an $S$-system, with the property, that either it has no zero element, or a unique zero element which is contained in any subsystem of $M$. If $M$ is fully (strongly) d-stable then it is duo.
Proof: By Remark (2.7), it is enough to prove the case (fully d-stable).
Let $M_S$ be a fully d-stable $S$-system, $f$ an endomorphism of $M_S$, and $N_S$ a subsystem of $M_S$. Then $\pi_N \circ f : M_S \to M_S / N_S$ is a homomorphism, which implies $N \times N \subseteq \ker \pi_N \circ f$. (where $\pi_N$ is the natural epimorphism). Let $x \in N$, we have two cases:

1. For all $y \in N$, $f(x) = f(y)$, then $f(N)$ is a one–element subsystem, and hence $f(x)$ is a fixed element of $M$ which must be unique and contained in any subsystem of $M$, (by hypothesis), that is, $f(x) \in N$.
2. There exists $y \in N$, with $f(x) \neq f(y)$, then $(x, y) \in \ker \pi_N \circ f$ implies $\pi_N(f(x)) = \pi_N(f(y))$, then $f(x), f(y) \in N$, therefore $f(x) \in N$ then, in the two cases $f(x) \in N \forall x \in N$, that is $f(N) \subseteq N$. So, $M_S$ is duo.

(2.10) **Proposition:** Every quasi-projective duo $S$-system is fully d-stable.

**Proof:**

Let $N$ be a subsystem of $M$, $\alpha : M \to M / N$ is a homomorphism, since $M$ is quasi-projective there exists an endomorphism $h$ of $M$ such that $\pi_N h = \alpha$ where $\pi_N$ is the natural epimorphism of $M$ onto $M / N$. Let $x, y \in N$ then $h(x) = xs, h(y) = yt$ for some $s, t \in S$ (since $M$ is duo) that is $(h(x), h(y))$ in $N \times N$ hence $\alpha(x) = \pi(h(x)) = \pi(xs) = \pi(yt) = \pi(h(y)) = \alpha(y)$ therefore $N \times N \subseteq \ker \alpha$.

(2.11) **Corollary:** Any duo monoid is fully d-stable.

**Proof:**
Let $V_S$ be a subsystem of $S$ and $\alpha : S \to S/V_S$ be a homomorphism, with $\pi_V$ surjective.

Assume that $\alpha(1) = [t]_V$, for some $t \in S$. Define $h$ by $h(s) = ts, \forall s \in S$,

$\pi_V(h(s)) = \pi_V(ts) = \pi_V(t)s = [t]_Vs = \alpha(1)s = \alpha(s)$ that is $\pi h = \alpha$, therefore $S_S$ is quasi-projective. Since $S_S$ is duo, then by (proposition 2.10), $S_S$ is fully d-stable. ■

**Theorem:** Let $M$ be a fully d-stable $S$-system and $N$ is a subsystem of $M$ such that $\alpha : M \to M/N$ is any $S$-homomorphism. Then for each $m \in M$ there exists $r \in S$ such that $\alpha(m) = [m]_N r$ (r depends on $\alpha$ and $m$).

**Proof:** Let $M_S$ be a fully d-stable $S$-system, and $N$ a subsystem of $M_S$, define $\lambda : M/N \to M/N$ by $\lambda([m]_N) = \alpha(m)$. For all $m_1, m_2 \in M$,

$[m_1] = [m_2] \Rightarrow m_1 = m_2$ or $m_1, m_2 \in N$(since $N \times N \subseteq \ker \alpha$), then $\alpha(m_1) = \alpha(m_2)$, hence $\lambda$ is well defined. It is clear that $\lambda$ is an $S$-homomorphism. Since $M/N$ is fully d-stable, it is duo (proposition 2.9). By( lemma 1.3) $\lambda([m]_N) = [m]r \Rightarrow \alpha(m) = [m]r$, for some $r \in S$. ■

**Corollary:** An $S$-system $M$ is fully d-stable if and only if for each subsystem $N$ each $S$-homomorphism $\alpha : M \to M/N$ has the property that for each $m \in M$, there exists $r \in S$ such that $\alpha(m) = [m]_N r$ (r depends on $\alpha$ and $m$).

**Proof:** By (Theorem 2.12).

$(\Leftarrow)$ Assume that for each $m \in M$, there exists $r \in S$ such that $\alpha(m) = [m]_N r$, let $m, n \in N$, then $\alpha(m) = [m]_N r = [mr]_N = \{N\}$ also $\alpha(n) = [n]_N s = [ns]_N = \{N\}$ for some $r, s \in S$ ($mr, ns \in N$), so $\alpha(m) = \alpha(n)$ then $(m, n) \in \ker \alpha$ that is $N \times N \subseteq \ker \alpha$. ■

**Proposition:** If $S$ is a monoid, then any free $S$-system with rank greater than one, is not fully d-stable.

**Proof:** Let $A_S$ be free $S$-system with rank more than one. Let $\{x_1, x_2\}$ be a subset of some basis $X$ of $A_S$ with $(x_1 \neq x_2)$.

Define $f : X \to S_S$ by

$$f(x) = \begin{cases} x & \text{if } x \notin \{x_1, x_2\} \\ x_1 & \text{if } x = x_1 \\ x_2 & \text{if } x = x_2 \\ x_1 & \text{if } x = x_2 \end{cases}$$

since $A_S$ is free $f$ can be extended to an endomorphism $\bar{f}$ of $A_S$.

Note that $\bar{f}(x_1) = x_2S \not\subseteq x_1S$, that is, $A_S$ is not duo, so by( proposition 2.9), $A_S$ cannot be fully d-stable. ■

**Proposition:** Let $S$ be a commutative monoid all of its elements satisfy left cancellation . If $A_S$ is a duo torsion free and indecomposable $S$-system then for all $f \in \text{End } A_S$, there exists $r \in S$ such that $f(a) = ar$ for all $a \in A$. (r depends only on $f$)

**Proof:** Assume $A_S$ is duo torsion free and $f : A_S \to A_S$, then for all $a \in A$ there exists $s \in S$ such that $f(a) = as$. Assume $a, b$ are distinct elements of $A_S$ and $(s, r \in S)$, $f(a) = as$ and $f(b) = br$. To prove $s = r$, $A_S \cap bS \neq \emptyset$, $\exists u, v \in S$ such that $au = bv$, $f(au) = asu$, $f(bv) = brv \Rightarrow asu = brv \Rightarrow au = bv$ ($S$ is commutative) $\Rightarrow s = r$. (since

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A_S is torsion free) therefore for all \( f : A_S \rightarrow A_S \), there exists \( r \) (depending on \( f \) only) such that \( f(a) = ar \) for all \( a \in A_S \).

**(2.16) Corollary:** Let \( S \) be a commutative monoid all of its elements satisfy left cancellation. If \( A_S \) is a duo torsion free and indecomposable \( S \)-system, then End \( A_S \cong S \) (as monoids).

**Proof:** \( \alpha : \text{End} \ A_S \rightarrow S \), for all \( f \in \text{End} \ A_S \), there exists unique \( r \) such that \( f(a) = ar \) for all \( a \in A_S \) by Proposition(2.15), define \( \alpha(f) = r \). Now, if \( f, g \in \text{End} \ A_S \), \( \alpha(f) = r, \alpha(g) = s \), then \( (gf)(a) = g(f(a)) = g(ar) = (ar)s = a(rs) \Rightarrow \alpha(gf) = rs \Rightarrow \alpha(gf) = \alpha(g) \alpha(f) \), hence \( \alpha \) is a monoid homomorphism. \( \alpha \) is onto, \( r \in S \), let \( f(a) = ar, f \in \text{End} \ A_S \). If \( \alpha(f) = \alpha(g) = r \Rightarrow \alpha(f) = \alpha(g) \alpha(f) = \alpha(g) \alpha(f) \), that is \( \alpha \) is one to one, therefore \( \alpha \) is an isomorphism.

**(2.17) Proposition:** Let \( S \) be a commutative monoid all its elements satisfy left cancellation. If \( M \) is fully \( d \)-stable \( S \)-system and \( N \) is a subsystem of \( M \) such that \( M/N \) is torsion free. Then for each homomorphism \( \alpha : M \rightarrow M/N \) there is \( r \in S \) such that \( \alpha(m) = [m]r \), for all \( m \in M \).

**Proof:** Recall, to the proof of Theorem (2.12), \( \lambda \) is an \( S \)-endomorphism of the torsion-free duo \( S \)-system \( M/N \). Then by proposition (2.16) there exists \( r \in S \) such that \( \lambda([m]) = [m]r \) for all \( [m]_N \in M/N \). Then \( \alpha(m) = \lambda([m]) = [m]r \).

The concepts of Hopfian and Co-Hopfian, where discussed for modules, see (Ozcan,ital..,2006). These concepts can be defined analogously in systems.

**(2.18) Definition:** Let \( M \) be an \( S \)-system. \( M \) is called Hopfian (Co-Hopfian) if every surjective (injective) endomorphism of \( M \) is an isomorphism.

**(2.19) Proposition:** Every fully \( d \)-stable \( S \)-system is Hopfian.

**Proof:** Let \( M \) be a fully \( d \)-stable \( S \)-system. \( f : M_S \rightarrow M_S \) surjective. then \( M_S \cong M_S/\ker f \), let \( \alpha : M_S \rightarrow M_S/\ker f \) be an isomorphism, hence \( \ker \alpha = \Delta_M \) since \( M_S \) is fully \( d \)-stable, that is \( \ker f \subseteq \ker \alpha = \Delta_M \) that is \( \ker f = \Delta_M \), (where \( \Delta_M = \{(x,x) \mid x \in S \} \)), therefore \( f \) is an isomorphism.

**(2.20) Example:** Let \( S = (N, \cdot)\), then \( N_S \) is a fully \( d \)-stable \( S \)-system (Remark 2.2), it is not Co-Hopfian since \( f : N_S \rightarrow N_S, f(n) = 2 n \) is an injective homomorphism, which is not isomorphism.

### 3. Dual stability and quasi-projective \( S \)-system

**(3.1) Proposition:** If \( N \) is fully invariant subsystem of a quasi-projective \( S \)-system \( M \) then \( M/N \) is likewise quasi-projective.
Proof: Let $K/N$ be a subsystem of $M/N$ and $\alpha : M/N \rightarrow (M/N)/(K/N)$.

Let $\beta : (M/N)/(K/N) \rightarrow M/K$ be an isomorphism. Then $\exists g : M \rightarrow M$ such that

$$\pi_K \circ g = \beta \circ \alpha \circ \pi_N \quad \Rightarrow \quad (\beta \pi \pi_N) \circ g = \beta \circ \alpha \circ \pi_N \Rightarrow \pi \pi_N \circ g = \alpha \circ \pi_N \quad \cdots (1)$$

Where, $(\pi_K : M \rightarrow M/K, \pi_N : M \rightarrow M/N, \pi : M/N \rightarrow (M/N)/(K/N))$.

Define $f : M/N \rightarrow M/N$ by $f([m]) = [g(m)]$. Note that $[m_1] = [m_2]$ implies $m_1 = m_2$ or $m_1, m_2 \in N$, hence $g(m_1) = g(m_2)$ or $g(m_1), g(m_2) \in N$ (N fully invariant), and so $[g(m_1)]_N = [g(m_2)]_N$. Therefore, $f$ is well defined, $f(\pi_N(m)) = \pi_N(g(m))$ for each $m \in M$, $\pi \circ f \circ \pi_N = \alpha \circ \pi_N$ for each $m \in M$, $\pi \circ f \circ \pi_N = \alpha \circ (\pi_N(m))$.

From (1) and (2) $\Rightarrow \pi \circ f \circ \pi_N = \alpha$ for each $m \in M/N \Rightarrow f \circ \pi_N = \alpha$. 

(3.2) Proposition (Kilp and Mikhalev, 2000): An $S$-system $B_S$ is a retract of an $S$-system $A_S$ if and only if there exists a subsystem $\hat{A}_S$ of $A_S$ and an epimorphism $h : A_S \rightarrow \hat{A}_S$ such that $B_S \cong \hat{A}_S$ and $h(\hat{a}) = \hat{a}$ for every $\hat{a} \in \hat{A}_S$.

(3.3) Lemma: If $N$, $L$ are two subsystems of an $S$-system $M$ and if $L$ is a subsystem of $N$, then there exists an epimorphism $\beta : M/L \rightarrow M/N$ with $(N/L \times N/L) \cup \Delta_{M/L} = \ker \beta$.

Proof: Let $\beta : M/L \rightarrow M/N$ be defined by $[m]_L \mapsto [m]_N$, $\beta$ is well defined, since $([m_1]_L = [m_2]_L$ hence $m_1, m_2 \in L$ (since $L$ subsystem of $N$) then $m_1, m_2 \in N$ therefore $[m_1]_N = [m_2]_N$). It is clear that $\beta$ is a homomorphism.

$$\ker \beta = \{(l, [m_2]_L) | \beta([m_1]_L) = \beta([m_2]_L) \}$$

$$= \{(l, [m_2]_L) | [m_1]_N = [m_2]_N \}$$

$$= \{(l, [m_2]_L) | m_1 = m_2 \text{ or } m_1, m_2 \in N \}$$

$$= (N/L \times N/L) \cup \Delta_{M/L}.$$
**Proposition:** Let N be a d-stable retract of an S-system M and L is a subsystem of N, then L is d-stable in N if and only if L is d-stable in M.

**Proof:** ($\Rightarrow$) Let $\alpha: M \to M/L$ be homomorphism and $\beta: M/L \to M/N$ be as in Lemma(3.3).

Note that $\ker \beta = (N/L \times N/L) \cup \Delta_{M/L}$. If for all $x, y \in N$, $\alpha(x) = \alpha(y)$, then $L \times L \subseteq N \times N \subseteq \ker \alpha$. Now, let $x \in N$, and there exists $y \in N$ such that

$\alpha(x) \neq \alpha(y)$ since $\beta \alpha: M \to M/N$, N is d-stable in M and $(x, y) \in N \times N \subseteq \ker \beta \alpha$, we have $\beta(\alpha(x)) = \beta(\alpha(y))$, so $(\alpha(x), \alpha(y)) \in \ker \beta$, this implies $\alpha(x) \in N/L$ for all $x \in N$, that is $\delta = \alpha|_N: N \to N/L$, by d-stability of L in N, $L \times L \subseteq \ker \delta = (\ker \alpha) \cap (N \times N)$, therefore $L \times L \subseteq \ker \alpha$. So, L is d-stable in M.

($\Leftarrow$) Let L be a d-stable in M and $\alpha: N \to N/L$ is an S-homomorphism, let $\beta$ be a homomorphism of M onto N, such that $\beta|_N = i_N$. (since N is a retract of M), by Definition(1.11). Then $\alpha \circ \beta: M \to M/L$, hence $L \times L \subseteq \ker \alpha \beta$

$= \{(x, y) \in M \times M| (\alpha \beta)(x) = (\alpha \beta)(y) \}$

$= \{(x, y) \in M \times M| \alpha(\beta(x)) = \alpha(\beta(y)) \}$

$(x, y) \in L \times L \Rightarrow x, y \in N \Rightarrow \beta(x) = x$ and $\beta(y) = y \Rightarrow \alpha(x) = \alpha(y) \Rightarrow (x, y) \in \ker \alpha$. 

Note that the retract property in Proposition (3.2) used only in the sufficient condition. So the following corollary clarifies the transitivity of d-stability.

**Corollary:** Let A, N and K be sub-systems of a system M with $A \subseteq N \subseteq K$. If A is d-stable in N and N is d-stable in K, then A is d-stable in K.

**Corollary:** A homomorphic image of a strongly d-stable quasi-projective S-system is likewise quasi-projective.

**Proof:** Let M be a strongly d-stable quasi-projective S-system, $\alpha: M \to \tilde{M}$ is an S-epimorphism.
First, note that if $h \in \text{End } M$, then $\alpha$ and $\alpha h$ are two homomorphisms from $M$ into $\hat{M}$, with $\alpha$ onto, by strong $d$-stability we have $\ker \alpha \subseteq \ker \alpha h$, that is, $\alpha(m) = \alpha(m')$ implies $\alpha(h(m)) = \alpha(h(m'))$.

Assume that $f$, $g : M \to A$ be homomorphisms, with $g$ onto, then $g \alpha$, $f \alpha$ are homomorphisms from $M$ into $A$ with $g \alpha$ onto, since $M$ is quasi-projective, there exists $h : M \to M$ such that $g \alpha h = f \alpha$. Define $\hat{h} : \hat{M} \to \hat{M}$ by $\hat{h}(x) = \alpha(h(m))$, where $m \in M$ with $x = \alpha(m)$ ($\alpha$ is onto), $\hat{h}$ is well defined by$(*), also$ $g \alpha h = g \alpha h = f \alpha$, but $\alpha$ is onto implies $g \hat{h} = f$. therefore $\hat{M}$ is quasi-projective.

References