Piecewise 3-Monotone Approximation on L_P-Spaces for < 1

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Abstract

In this paper we introduce a Jackson type theorem for piecewise 3-monotone approximation of function in L_P-spaces for $P < 1$.

Keywords: 3-Monotone approximation by piecewise polynomials; Degree of approximation

الخلاصة

قدمنا في هذا البحث نوعا من مبرهنة جاكسون للتقريب المتقطع للدوال رتيبة-3 للفضاءات L_P ,$P < 1$

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1.Introduction

Let $f: [a, b] \to R$ is called K-monotone , $K \geq 1$ on $[a, b]$, if $f[x_0, x_1, x_2, \dots, x_K] \geq 0$ for each choices of $K + 1$ distinct point $x_0, x_1, x_2, \dots, x_K$ in $[a, b]$.

Denote by

$$f[x_0, \dots \dots, x_K] = \sum_{u=0}^{K} \frac{f(x_u)}{\prod_{v=0,\ v\neq u}^{K} (x_u - x_v)}$$

denotes the Kth divide difference of f at the distinct point $x_0, x_1, x_2, \dots, x_K$. We denote that $\Delta^1_{[a,b]}$ is nondecreasing function on $[a, b]$ and $\Delta^2_{[a,b]}$ is convex function on $[a, b]$. Note that $\Delta^3_{[a,b]}$ is 3-monotone on $[a, b]$.

Higher order shape preserving approximation ,i.e K-monotone approximation , had been investigated in the results, many positive and negative results for K-monotone approximation $K > 2$, had been introduce which goes back to (Shvedov, 1981), the results "if $f \in \Delta^K_{[-1,1]} \cap L^{(K)}_{P[-1,1]}$, and $0 \leq f^{(K)}(x) \leq 1$, $\forall\, x \in [-1,1]$, there exist apiecewise polynomial $s \in \Delta^K_{[-1,1]}$ of degree $\leq K - 1$ with distant n such that $\|f(x) - s(x)\|_{L_P[-1,1]} \leq c(K)n^{-K}$, $x \in [-1,1]$ ".

If $F \in \Delta^3_{[a,b]} \cap L^{(2)}_{P[a,b]}$ Konovalov and Leviatan [Konovalov & Leviatan, 2001] have constructed 3-monotone quardratic spline S satisfying

$$\|F - S\|_{L_P[a,b]} \leq \frac{c}{n^2} \omega_1(F'';\frac{1}{n})$$

Where $c = c(a, b)$ is abslute constant independed of F and .

In [Shvedov, 1981] Prymak extended the theorem of [Konovalov & Leviatan 2003; Leviatan& Prymak, 2005] and gave on estimate for the degree of approximation in terms of the third-modulus of smoothness . As a consequence of the above result is the following for all $F \in \Delta^3_{[a,b]} \cap L_P[a,b]$ there is a piecewise quadratic $S \in \Delta^3_{[a,b]} \cap L_P[a,b]$ and with equidistant knot n . which

$$\|F - S\|_{L_P[a,b]} \leq c\omega_3(F; \frac{1}{n})$$

since $c = c(a,b)$ is abslute constant **.**

The main aim of our paper is to answer the following question: Can we achieve a higher order estimate for the degree of best approximation of function s in L_P , $P < 1$ by 3-monotone piecewise polynomials?

In this paper we generalize the results in [Beatson, 1981; Bondarenko, 2002] to functions in L_P-spaces , $P < 1$, i.e we prove

Theorem 3.1 : let F be a 3-monotone function in $L_P[a,b]$, $P < 1$, satisfying $f(x) = F'(x)$, for each $x \in (a,b)$, and let $k \geq 1$, $a = x_0 < x_1 < \cdots < x_n = b$, be apartition for the interval $[a,b]$.Also let s be convex piecewise polynomial on $[a,b]$ of degree $\leq k-1$ interpolating f at knots x_u , $u = 1,2,\ldots,n-1$, such that

$$s(x_u) = f(x_u) \tag{1}$$

Then there exist a 3-monotone polynomial in $L_P[a,b]$, P of degree $\leq k$ with the same knots and satisfy

$$\|F - P\|_{L_P[a,b]} \leq c(p)\|f - s\|_{L_P[x_{u-1},x_u]} \tag{2}$$

$c(p)$ is an absolute constant depending on p only.

Remark 1.1 [Leviatan& Prymak, 2005**] :**$\Delta^3_{[a,b]}$ is the set of all bounded function, having a convex derivative on (a,b).Note that if $f \in \Delta^k_{[a,b]}$,$k \geq 2$,then f is continous on (a,b) and $f(a+)$, $f(b-)$ exist and are finite.

Defintion1.2[Leviatan& Prymak, 2005**]:**let f , g real functions in $L_P[a,b]$ .Then

$$\|f - g\|_{L_P[a,b]} \leq c(p)(\|f\|_{L_P[a,b]} + \|g\|_{L_P[a,b]})$$

and $\|.\|_{L_P[a,b]}$ is called quasinormed space L_P**,** $P < 1$ and $c(p)$ is an absolute constant depending on P **.**

2. The auxiliary result

In this section we introduce the lemmas that we need to prove our theorem.

Lemma 2.1[Leviatan& Prymak, 2005**]:** let $f \in \Delta^{\mathbf{2}}_{[\boldsymbol{a},\boldsymbol{b}]}$ and suppose that $q \in \Delta^2_{[a,b]}$ is a polynomial of degree $\leq k-1$ satisfying $f(a) = q(a)$ and $f(b) = f(b)$. Then there exists apolynomial $p \in \Delta^2_{[a,b]}$ of degree $\leq k-1$ such that

(1) $p(x) = \frac{Al(x)+Bq(x)}{A+B}$,

Where $A = \int_a^b f(t)dt - \int_a^b f(t)dt$, $B = \int_a^b l(t)dt - \int_a^b f(t)dt$

and l be the Lagrange polynomial of f at a and b.

(2)$f(a) = p(a)$, $f(b) = p(b)$

$q'(a) \leq p'$, $p'(b) \leq q'(b)$

(3)$\int_a^b p(x)dx \geq \int_a^b f(x)dx$.

Lemma 2.2: let $f \in \Delta^2_{[a,b]} \cap Lp[a,b], p < 1$ and assume q is a convex polynomial of degree $\leq k-1$ in $L_P[a,b]$ interpolation f at a and b .Then there is a convex polynomial p in $L_P[a,b]$ of degree $\leq k-1$, satisfying
(1) interpolation f at a and b.
(2)$q'(a) \leq p'(a)$, $p'(b) \leq q'(b)$.
(3)$\left\|\int_a^{(.)}\big(p(t)-f(t)\big)dt\right\|_{L_P[a,b]} \leq c(p)\left\|\int_a^{(.)}\big(q(t)-f(t)\big)dt\right\|_{L_P[a,b]}$
(4)$\int_a^b p(t)dt \geq \int_a^b f(t)dt$.

Proof: by Lemma 2.1 we get (1) ,(2) and (4)

And by Lemma 2.1 ,

Let $B = \int_a^b l(t)dt - \int_a^b f(t)dt$,

$$A = \int_a^b f(t)dt - \int_a^b f(t)dt$$

and

$$p(x) = \frac{Al(x) + Bq(x)}{A+B}$$

To prove (3),

$$\left\|\int_a^x p(t)dt - \int_a^x f(t)dt\right\|_{L_P[a,b]} = \left\|\int_a^x \left(\frac{Al(t) - Bq(t)}{A+B}\right)dt - \int_a^x f(t)dt\right\|_{L_P[a,b]}$$

$$= \left\|\frac{A}{A+B}\int_a^x \big(l(t)-f(t)\big)dt + \frac{A}{A+B}\int_a^x \big(q(t)-f(t)\big)dt\right\|_{L_P[a,b]}$$

$$\leq c(p)\left(\frac{A}{A+B}\left\|\int_a^x \big(l(t)-f(t)\big)dt\right\|_{L_P[a,b]} + \frac{B}{A+B}\left\|\int_a^x \big(q(t)-f(t)\big)dt\right\|_{L_P[a,b]}\right)$$

$$\leq c(p)\left(\frac{A}{A+B}B + \frac{B}{A+B}\left\|\int_a^{(.)} \big(q(t)-f(t)\big)dt\right\|_{L_P[a,b]}\right)$$

$$\leq c(p)\frac{B}{A+B}\left\|\int_a^{(.)} \big(q(t)-f(t)\big)dt\right\|_{L_P[a,b]} + \frac{B}{A+B}\left\|\int_a^{(.)} \big(q(t)-f(t)\big)dt\right\|_{L_P[a,b]}$$

$$\leq \mathrm{c(p)} \left\| \int_{\mathrm{a}}^{(.)} (\mathrm{q(t)} - \mathrm{f(t)}) \mathrm{dt} \right\|_{L_P[a,b]} \quad \blacksquare \tag{3}$$

Lemma 2.3 [Leviatan& Prymak, 2005**]:**let $q \in \Delta^2_{[a,b]}$ be polynomial of degree $\leq n-1$, and let α and β arbitrary non-negative real numbers . Suppose that d_a , d_b are real numbers satisfying ,

$$d_a \leq \frac{(q(b)-\beta)-(q(a)-\alpha)}{b-a} \leq d_b \tag{4}$$

And $d_a \leq q^{'}(a) \leq q^{'}(b) \leq d_a$. Then there exists apolynomial $p \in \Delta^2_{[a,b]}$ of degree $\leq n-1$, such that

$$p(a) = q(a) - \alpha \ , \ p(b) = q(b) - \beta \tag{5}$$

$$d_a \leq p^{'}(a) \leq p^{'}(b) \leq d_a \tag{6}$$

and

$$p(x) \leq q(x) \ , \ x \in [a,b] \tag{7}$$

Lemma 2.4: let $f\ ,g\ \in \Delta^2_{[z_1,z_2]}\ \cap L_P[z_1,z_2]$, <1 , satisfying $f(z_u) = g(z_u)$ and $l_u(x) = (x - z_u)g^{'}(z_u) + g(z_u)$, $u = 1,2$

Then

$$\int_{z_1}^{z_2} (l_u(t) - f(t))_+ \ dt \ \leq c_1 c_2 \|g(t) - f(t)\|_{L_P[z_1,z_2]}$$

Proof: if $u = 1$, let g is convex , and $l_1 \leq g(x)$, for each $x \in [z_1, z_2]$. Let f is convex and l_u is linear , there is a $\theta \in [z_1, z_2]$, so that $f(x) \leq l_1(x)$, for each $x \in [z_1, \theta]$, such that $l_1(x) = f(x)$ **,** for each $x \in [\theta, z_2]$.

if $u = 1$

$$\int_{z_1}^{\theta} (g(t) - f(t)) dt = \left(\int_{z_1}^{\theta} (g(t) - f(t))^{1-p+p} dt \right)^{1-\frac{1}{p}+\frac{1}{p}}$$

$$= \left(\int_{z_1}^{\theta} (g(t) - f(t))^{1-p} (g(t) - f(t))^p \right)^{\frac{1}{p}} \left(\int_{z_1}^{\theta} (g(t) - f(t))^{1-p} (g(t) - f(t))^p \right)^{1-\frac{1}{p}} dt$$

$$\leq c_1 c_2 \left(\int_{z_1}^{\theta} (g(t) - f(t))^p dt \right)^{\frac{1}{p}} \ ,$$

Where $c_1 = \left(\int_{z_1}^{\theta} (g(t) - f(t))^{1-\ p} \right)^{\frac{1}{p}}$,

and $c_2 = \left(\int_{z_1}^{\theta} (g(t) - f(t))^{1-p} (g(t) - f(t))^p \right)^{1-\frac{1}{p}}$

$$\int_{z_1}^{\theta}\big(g(t)-f(t)\big)dt \le c_1c_2\|g-f\|_{L_P[Z_1,\theta]}$$

if $u=2$

$$\int_{\theta}^{z_2}\big(g(t)-f(t)\big)dt = \left(\int_{\theta}^{z_2}(g(t)-f(t))^{1-p+p}dt\right)^{1-\frac{1}{p}+\frac{1}{p}}$$

$$= \left(\int_{\theta}^{z_2}\big(g(t)-f(t)\big)^{1-p}(g(t)-f(t))^{p}\right)^{\frac{1}{p}}\left(\int_{\theta}^{z_2}\big(g(t)-f(t)\big)^{1-p}(g(t) - f(t))^{p}\right)^{1-\frac{1}{p}}dt$$

$$\le c_1c_2\big(\textstyle\int_{\theta}^{z_2}(g(t)-f(t))^{p}dt\big)^{\frac{1}{p}} ,$$

where $c_1 = \left(\int_{\theta}^{z_2}\big(g(t)-f(t)\big)^{1-p}\right)^{\frac{1}{p}}$,

and

$$c_2 = \left(\textstyle\int_{\theta}^{z_2}\big(g(t)-f(t)\big)^{1-p}(g(t)-f(t))^{p}\right)^{1-\frac{1}{p}}$$

$$\int_{\theta}^{z_2}\big(g(t)-f(t)\big)dt \le c_1c_2\|g-f\|_{L_P[\theta,z_2]}$$

Hence ,

$$\textstyle\int_{z_1}^{z_2}\big(l_u(t)-f(t)\big)_+ dt \le c_1c_2\|g(t)-f(t)\|_{L_P[z_1,z_2]} \qquad \blacksquare$$

Lemma 2.5 [Leviatan& Prymak, 2005**]:** let f , g $\in \Delta^2_{[a,b]}$ be such that

$$f(b)-f(a) = g(b)-g(a) \tag{8}$$

Then

$$f'(a+) \le g'(b-)$$

Corollary 2.6 [Leviatan& Prymak, 2005**]:** let f $\in \Delta^2_{[a,b]}$ and s $\in \Delta^2_{[a,b]}$ be a piecewise polynomial of degree $\le n-1$, with knots on the partition $a = x_0 < x_1 < \cdots < x_n = b$, satisfying

$s(x_u) = f(x_u)$. Then for $u = 2, \dots, n-1$

$$f'(x_{u-1}+) \le s'(x_u-) \ , \ u = 2, \dots, n-1 \tag{9}$$

and

$$s'(x_{u-1}+) \le f'(x_u -) \ , \ u = 2, \dots, n-1 \tag{10}$$

Remark 2.7: we are begin our auxiliary construction . Let f , s are convex a piecewise polynomial on $[a,b]$ of degree $\le n-1$. Then the function g , we write $g \in A_{u,v}$, $u \le v \le n-1$, and g is convex a piecewise polynomial on $[x_u, x_v]$ of degree $\le n-1$, with knots $x_{u+1}, \dots, x_{v-1}$, and satisfies $s'(x_u +) \le g'(x_u +)$ and $g'(x_v -) \le s'(x_v -)$ and $g(x_u) = s(x_u)$ and $g(x_v) = s(x_v)$, $\forall\, r = 1, \dots, n-1$. [2]

Let

$$h_r(t) = \begin{cases} f'(x_u -) & IF \quad t \in (x_{u-1}, x_u], \quad u = 1,2,\dots,n-1 \\ s'(x_r -) & IF \quad t \in (x_{r-1}, x_u], \quad u = 1,2,\dots,n-1 \\ s'(x_r +) & IF \quad t \in (x_r, x_{r+1}), \quad u = 1,2,\dots,n-1 \\ f'(x_{i-1} +) & IF \quad t \in [x_{u-1}, x_u), \quad u = 1,2,\dots,n-1 \end{cases} \qquad [2]$$

and

let $g_r(x) = f(x_r) + \int_{x_r}^{x} h_r(t)dt$ [Leviatan& Prymak, 2005]

By virtue of (Corollary 2.6 **)**, h_r is non-decreasing on (a,b) , and g_r is convex . It follow by $s(x_u) = f(x_u)$ that $g_r(x_{r+1}) \le f(x_{r+1})$ and $g_r(x) \le f(x_{r-1})$ .Hence,

$g_r(x) \le f(x)$, $x \in [a,b] \backslash (x_{r-1}, x_{r+1})$ [Leviatan& Prymak, 2005] (11)

By Lemma 2.4,

$$\int_{x_r}^{x_{r+1}} \big(g_r(t) - f(t)\big)dt = \left(\int_{x_r}^{x_{r+1}} (g_r(t) - f(t))^{1-p+p} dt\right)^{1-\frac{1}{p}+\frac{1}{p}}$$

$$= \left(\int_{x_r}^{x_{r+1}} \big(g_r(t) - f(t)\big)^{1-p} (g(t) - f(t))^p\right)^{\frac{1}{p}} \left(\int_{x_r}^{x_{r+1}} \big(g_r(t) - f(t)\big)^{1-p} (g_r(t) - f(t))^p\right)^{1-\frac{1}{p}} dt$$

$$\le c_1 c_2 \left(\int_{x_r}^{x_{r+1}} (g_r(t) - f(t))^p dt\right)^{\frac{1}{p}} ,$$

where $c_1 = \left(\int_{x_r}^{x_{r+1}} \big(g_r(t) - f(t)\big)^{1-p}\right)^{\frac{1}{p}}$

and

$$c_2 = \left(\int_{x_r}^{x_{r+1}} \big(g_r(t) - f(t)\big)^{1-p} (g_r(t) - f(t))^p\right)^{1-\frac{1}{p}}$$

$$\int_{x_r}^{x_{r+1}} \big(g_r(t) - f(t)\big)dt \le c_1 c_2 \|g - f\|_{L_P[x_r, x_{r+1}]} \qquad (12)$$

And

$$\int_{x_{r-1}}^{x_r} \big(g_r(t) - f(t)\big)dt = \left(\int_{x_{r-1}}^{x_r} (g_r(t) - f(t))^{1-p+p} dt\right)^{1-\frac{1}{p}+\frac{1}{p}}$$

$$= \left(\int_{x_{r-1}}^{x_r} \left(g_r(t) - f(t)\right)^{1-p} (g(t) - f(t))^p \right)^{\frac{1}{p}} \left(\int_{x_{r-1}}^{x_r} \left(g_r(t) - f(t)\right)^{1-p} (g_r(t) - f(t))^p \right)^{1-\frac{1}{p}} dt$$

$$\leq c_1 c_2 \left(\int_{x_{r-1}}^{x_r} (g_r(t) - f(t))^p dt \right)^{\frac{1}{p}} ,$$

Where

$$c_1 = \left(\int_{x_{r-1}}^{x_r} \left(g_r(t) - f(t)\right)^{1-p} \right)^{\frac{1}{p}}$$

and

$$c_2 = \left(\int_{x_{r-1}}^{x_r} \left(g_r(t) - f(t)\right)^{1-p} (g_r(t) - f(t))^p \right)^{1-\frac{1}{p}}$$

Hence,

$$\int_{x_{r-1}}^{x_r} \left(g_r(t) - f(t)\right) dt \leq c_1 c_2 \|g - f\|_{L_P[x_{r-1}, x_r]} \tag{13}$$

Lemma 2.8 [Leviatan .D., Prymak .A.V.**]:**

$\int_{x_{m-1}}^{x_m} (g_{u,v}(t) - f(t))_+ dt \leq \int_{x_{m-1}}^{x_m} (s(t) - f(t))_+ dt$, for each $x \in [x_{m-1}, x_m]$

Where $s \in \Delta^2_{[a,b]}$ be a piecewise polynomial of degree $\leq n-1$.

Remark 2.9: by virtue of (12) ,(13) we have

$$\int_{x_u}^{x_{u+1}} \left(g_{u,v}(t) - f(t)\right) dt = \left(\int_{x_u}^{x_{u+1}} (g_{u,v}(t) - f(t))^{1-p+p} dt \right)^{1-\frac{1}{p}+\frac{1}{p}}$$

$$= \left(\int_{x_u}^{x_{u+1}} \left(g_{u,v}(t) - f(t)\right)^{1-p} (g(t) - f(t))^p \right)^{\frac{1}{p}} \left(\int_{x_u}^{x_{u+1}} \left(g_{u,v}(t) - f(t)\right)^{1-p} (g_{u,v}(t) - f(t))^p \right)^{1-\frac{1}{p}} dt$$

$$\leq c_1 c_2 \left(\int_{x_v}^{x_{u+1}} (g_{u,v}(t) - f(t))^p dt \right)^{\frac{1}{p}} ,$$

where $c_1 = \left(\int_{x_u}^{x_{u+1}} \left(g_{u,v}(t) - f(t)\right)^{1-p} \right)^{\frac{1}{p}}$

and

$$c_2 = \left(\int_{x_u}^{x_{u+1}} \left(g_{u,v}(t) - f(t)\right)^{1-p} (g_{u,v}(t) - f(t))^p \right)^{1-\frac{1}{p}}$$

$$\int_{x_u}^{x_{u+1}} \left(g_{u,v}(t) - f(t)\right) dt \leq c_1 c_2 \|g - f\|_{L_P[x_u , x_{u+1}]} \qquad (14)$$

And

$$\int_{x_{v-1}}^{x_v} \left(g_{u,v}(t) - f(t)\right) dt = \left(\int_{x_{v-1}}^{x_v} (g_{u,v}(t) - f(t))^{1-p+p} dt\right)^{1-\frac{1}{p}+\frac{1}{p}}$$

$$= \left(\int_{x_{v-1}}^{x_v} \left(g_{u,v}(t) - f(t)\right)^{1-p} (g(t) - f(t))^p\right)^{\frac{1}{p}} \left(\int_{x_{v-1}}^{x_v} \left(g_{u,v}(t) - f(t)\right)^{1-p} (g_{u,v}(t) - f(t))^p\right)^{1-\frac{1}{p}} dt$$

$$\leq c_1 c_2 \left(\int_{x_{v-1}}^{x_v} \left(g_{u,v}(t) - f(t)\right)^p dt\right)^{\frac{1}{p}},$$

where $c_1 = \left(\int_{x_{v-1}}^{x_v} \left(g_{u,v}(t) - f(t)\right)^{1-p}\right)^{\frac{1}{p}}$

and $c_2 = \left(\int_{x_{v-1}}^{x_v} \left(g_{u,v}(t) - f(t)\right)^{1-p} (g_{u,v}(t) - f(t))^p\right)^{1-\frac{1}{p}}$

$$\int_{x_{v-1}}^{x_v} \left(g_{u,v}(t) - f(t)\right) dt \leq c_1 c_2 \|g - f\|_{L_P[x_{v-1} , x_v]}$$
(15)

Lemma 2.10 : let $g_{u,v}(x) \leq f(x)$, $\forall\, x \in [x_{l-1}, x_l]$,and $u + 1 < l < v$, $l \neq m$ and $g_{u,v}$ is a convex polynomial of degree $\leq k - 1$

Then

$$\int_{x_u}^{x_v} (g_{u,v}(t) - f(t))_+ \, dt \leq 3c_1 c_2 \|g - f\|_{L_P[x_u , x_v]}$$

Proof: from Lemma 2.8 ,(14)and(15) ,we have

$$\int_{x_u}^{x_v} (g_{u,v}(t) - f(t) dt \leq \int_{x_u}^{x_{u+1}} \left(g_{u,v}(t) - f(t)\right) dt + \int_{x_{m-1}}^{x_m} \left(g_{u,v}(t) - f(t)\right) dt + \int_{x_{v-1}}^{x_v} \left(g_{u,v}(t) - f(t)\right) dt$$

$$\leq c_1 c_2 \|g - f\|_{L_P[x_u , x_v]} + c_1 c_2 \|g - f\|_{L_P[x_u , x_{u+1}]} + c_1 c_2 \|g - f\|_{L_P[x_v , x_{v-1}]}$$

Where c_1 and c_2 as we define above.

$$\int_{x_u}^{x_v} (g_{u,v}(t) - f(t))\, dt \leq 3c_1 c_2 \|g - f\|_{L_P[x_u , x_v]} \qquad \blacksquare$$

Lemma 2.11: let $\delta(.) \in L_P[a,b], P < 1$,then we have

$$\left\|\int_{x_u}^{(.)}\delta(t)dt\right\|_{L_P[x_u,x_v]} \leq \left\|\int_{x_u}^{x_v}\delta(t)dt\right\|_{L_P[x_u,x_v]} + \left\|\int_{x_u}^{x_v}\delta(t)_+dt\right\|_{L_P[x_u,x_v]}$$

Proof: let $x_u \leq x \leq x_v$ if $\int_{x_u}^{x}\delta(t)dt \geq 0$, then

$$0 \leq \left\|\int_{x_u}^{x}\delta(t)_+dt\right\|_{L_P[x_u\,,x]} \leq \left\|\int_{x_u}^{x}\delta_+(t)dt\right\|_{L_P[x_u,x]} \leq \left\|\int_{x_u}^{x_v}\delta_+(t)dt\right\|_{L_P[x_u,x_v]}$$

And if $\int_{x_u}^{x}\delta(t) < 0$,

$$\left\|\int_{x_u}^{x}\delta(t)dt\right\|_{L_P[x_u,x]} \leq \left\|\int_{x_u}^{x}\delta_-(t)dt\right\|_{L_P[x_u,x]} \leq \left\|\int_{x_u}^{x_v}\delta_-(t)dt\right\|_{L_P[x_u,x_v]}$$

$$\leq \left\|-\int_{x_u}^{x_v}\delta(t)dt\right\|_{L_P[x_u\,,x_v]} + \left\|\int_{x_u}^{x_v}\delta(t)dt\right\|_{L_P[x_u\,,x_v]}$$

Hence $\left\|\int_{x_u}^{(.)}\delta(t)dt\right\|_{L_P[x_u,x_v]} \leq \left\|\int_{x_u}^{x_v}\delta(t)dt\right\|_{L_P[x_u\,,x_v]} + \left\|\int_{x_u}^{x_v}\delta_+(t)dt\right\|_{L_P[x_u,x_v]}$

Then by Lemma 2.10 , we have

$\left\|\int_{x_u}^{(.)}(g_{u,v}(t) - f(t))dt\right\|_{L_P[x_u,x]} \leq \left\|\int_{x_u}^{x_v}\left(g_{u,v}(t) - f(t)\right)dt\right\|_{L_P[x_u\,,x_v]} + 3c_1c_2\|g - f\|_{L_P[x_u\,,\,x_v]}$ ■ (16)

Lemma 2.12 : The fixed integer $1 \leq u \leq n-2$. Then there is $u+1 \leq v \leq n-1$, and afunction $g_{u,v}^* \in A_{u,v}$ and $A_{u,v}$ is a convex piecewise polynomial of degree $\leq k-1$

such that

$$\left\|\int_{x_u}^{(.)}\left(g_{u,v}^*(t) - f(t)\right)dt\right\|_{L_P} \leq 6c_1c_2\|g - f\|_{L_P} \tag{17}$$

If $v \leq n-1$, then

$$\int_{x_u}^{x_v}(g_{u,v}^*(t) - f(t))dt \leq 0 \tag{18}$$

Proof : If $\int_{x_u}^{x_{n-1}}\left(g_{u,v}(t) - f(t)\right)dt \geq 0$

Then by Lemma 2.10 ,

$$\int_{x_u}^{x_{n-1}}\left(g_{u,v}(t) - f(t)\right)dt \leq 3c_1c_2\|g - f\|_{L_P[x_u\,,\,x_v]}$$

and put $g_{u,n-1}^* = g_{u,n-1}$

(17) follows by (16)

Otherwise, at smallest of the above numbers $\int_{x_u}^{x_{u+r}}(g_{u,v}(t)-f(t))dt$, $u \le r \le n-1$ is anegative .

Let $u \le r \le n-u-1$, $-6c_1c_2\|g-f\|_{L_P[x_u\,,\,x_v]} \le \int_{x_u}^{x_{u+r}}\left(g_{u,v}(t)-f(t)\right)dt < 0$

We put $v = u+r$ so that $g_{u,v}^{*} = g_{u,v}$,then (18) is complete , and by (16) ,We get (17).

Finally,if all numbers are anegative among the above are $< -6c_1c_2\|g-f\|_{L_P}$, then let $1 \le r \le n-u-1$,be the little such that

$$\left\|\int_{x_u}^{x_{u+r}}\left(g_{u,v}(t)-f(t)\right)dt\right\|_{L_P} \le -6c_1c_2\|g-f\|_{L_P} \ , \ \ \mathrm{r} \ge 2$$

since $g_{u,u+1}(x) = s(x)$,for all $x \in [x_u\,, x_{v+1}]$,

whence $\left\|\int_{x_u}^{x_{u+1}}\left(g_{u,v}(t)-f(t)\right)dt\right\|_{L_P} \le c_1c_2\|g-f\|_{L_P}$,

Put $v = u+r$, and $p = s_{|_{[x_{v-1},x_v]}}$,

Then

$$\left\|\int_{x_{v-1}}^{(.)}\left(p(t)-f(t)\right)dt\right\|_{L_P[x_{v-1}\,,x_v]} \le c_1c_2\|g-f\|_{L_P} \tag{19}$$

Let

$$\ddot{g}_{u,v}(x) = \begin{cases} g_{u,v-1}(x) & IF \quad x \in [x_u\,,\, x_{v-1}\,) \\ p(x) & IF \quad x \in [x_{v-1}\,,\, x_v\,] \end{cases}$$

Then $\ddot{g}_{u,v} \in A_{u\,,v-1}$ and $\ddot{g}_{u,v} \in A_{v-1\,,v}$, such that $\ddot{g}_{u,v} \in A_{u\,,v}$. Let $g_{u,v}^{*}(x) = \lambda\, g_{u,v}(x) + (1-\lambda)\ddot{g}_{u,v}(x)$, for all $x \in [x_u\,, x_v]$,

where $\lambda = 6c_1c_2\|g-f\|_{L_P}/\left\|\int_{x_u}^{x_v}\left(g_{u,v}(t)-f(t)\right)dt\right\|_{L_P} = 5\,c_1c_2\|g-f\|_{L_P}$, $\lambda > 0$

So that $g_{u,v}^{*} \in A_{u\,,v}$ .The select r implies that $0 \le \int_{x_u}^{x_{v-1}}\left(g_{u,v}(t)-f(t)\right)dt \le 3c_1c_2\|g-f\|_{L_P}$

then by (16)

$$\left\|\int_{x_u}^{(.)}(\ddot{g}_{u\,,v}(t)-f(t))dt\right\|_{L_P[x_u,x]} = \left\|\int_{x_u}^{x_{v-1}}(\ddot{g}_{u\,,v}(t)-f(t))dt\right\|_{L_P[x_u,x_{v-1}]}$$

$$\le \left\|\int_{x_u}^{x_{v-1}}(\ddot{g}_{u\,,v}(t)-f(t))dt\right\|_{L_P} + 3c_1c_2\|g-f\|_{L_P}$$

$$\le 3c_1c_2\|g-f\|_{L_P} + 3c_1c_2\|g-f\|_{L_P}$$

$$\le 6c_1c_2\|g-f\|_{L_P}$$

Also ,by (19)

$$\left\|\int_{x_u}^{x}(\ddot{g}_{u,v}(t)-f(t))dt\right\|_{L_P} = \left\|\int_{x_u}^{x_{v-1}}\left(\ddot{g}_{u,v}(t)-p(t)\right)dt+\int_{x_{v-1}}^{x}(p(t)-f(t))dt\right\|_{L_P}$$

$$\leq 2^{p-1}(\left\|\int_{x_u}^{x_{v-1}}\left(\ddot{g}_{u,v}(t)-f(t)\right)dt\right\|_{L_P}+\left\|\int_{x_{v-1}}^{x}(p(t)-f(t))dt\right\|_{L_P})$$

$$\leq 2^{p-1}(3c_1c_2\|g-f\|_{L_P}+c_1c_2\|g-f\|_{L_P})$$

$$\leq 4(2^{p-1}c_1c_2\|g-f\|_{L_P[x_{v-1},x_v]}) \qquad , \quad x\in[x_{v-1},x_u]$$

$$\leq 2^{p+1}c_1c_2\|g-f\|_{L_P}$$

Therefore ,

$$\left\|\int_{x_u}^{(.)}(\ddot{g}_{u,v}(t)-f(t))dt\right\|_{L_P[x_u,x_v]} \leq 6c_1c_2\|g-f\|_{L_P} \qquad (20)$$

In particular

$$\int_{x_u}^{x_v}\left(g_{u,v}(t)-f(t)\right)dt$$

$$=\lambda\int_{x_u}^{x_v}\left(g_{u,v}(t)-f(t)\right)dt+(1-\lambda)\int_{x_u}^{x_v}\left(\ddot{g}_{u,v}(t)-f(t)\right)dt$$

$$\leq 5\left(-6c_1c_2\|g-f\|_{L_P}\right)+(1-5)6c_1c_2\|g-f\|_{L_P}<0$$

We get (18) .Then by (16)and (20) ,

$$\left\|\int_{\mathrm{x_u}}^{\mathrm{x_v}}\left(\mathrm{g_{u,v}(t)-f(t)}\right)\mathrm{dt}\right\|_{\mathrm{l_p[x_u,x_v]}}$$

$$\leq\lambda\left\|\int_{\mathrm{x_u}}^{\mathrm{x_v}}\left(\mathrm{g_{u,v}(t)-f(t)}\right)\mathrm{dt}\right\|_{\mathrm{l_p[x_u,x_v]}}+(1$$

$$-\lambda)\left\|\int_{\mathrm{x_i}}^{(.)}(\ddot{\mathrm{g}}_{\mathrm{u,v}}(\mathrm{t})-\mathrm{f(t))dt}\right\|_{\mathrm{l_p[x_u,x_v]}}$$

$$\leq\lambda\left\|\int_{x_u}^{x_v}\left(g_{u,v}(t)-f(t)\right)dt\right\|_{L_P[x_u,x_v]}+3c_1c_2\|g-f\|_{L_P})+(1-\lambda)6c_1c_2\|g-f\|_{L_P}$$

$$\leq 5\left(6c_1c_2\|g-f\|_{L_P}\right)+6(1-5)c_1c_2\|g-f\|_{L_P}$$

$\leq 6c_1c_2\|g-f\|_{L_P}$ this proves (17) ■

3. The main result

In this section we introduce our main result

3.1 :The proof of theorem.

For the proof we follow the same strategies used in [Leviatan& Prymak, 2005] .

We search for the require function P in the

$P(x) = F(x_u) + \int_{x_1}^{x} \hat{g}(t)dt \quad , \quad \forall\, x \in [a, b]$

Let

$$\hat{g}(t) = \begin{cases} s(t) & if \quad t \in [x_0, x_1) \cup (x_{n-1}, x_n] \\ g(t) & if \quad t \in [x_1\,, x_{n-1}] \end{cases}$$

In $A_{1,n-1}$. The last we shall construct the 3-monotone polynomial of P .Construct g by induction.

Apply Lemma 2.2 on interval $[x_{u-1}, x_u]$, $2 \le u \le n-1$, and $q = s_{\mid [x_{u-1}, x_u]}$, then the polynomial p is in $A_{u-1,u}$. Also, recall if $g \in A_{u.,v}$, $1 \le u < v < l \le n-1$, and $g \in A_{u.,v}$, such that $g \in A_{u,l}$. We get construct g by induction. Apply Lemma 2.2 on interval $[x_1\,, x_2]$, with $q = s_{\mid [x_1\,, x_2]}$ we obtain on apolynomial $p \in A_{1,2}$, and take $p(x) = g(x)$. Assume that g is define on $[x_1\,, x_2]$, $1+1 \le u \le n-1-1$, $g \in A_{\,1,u}$, and satisfying for each $x \in [x_1\,, x_u]$,

$$\left\| \int_{x_1}^{x} \big(g(t) - f(t)\big)dt \right\|_{L_P[x_1\,, x]} \le 12c_1c_2 \|g - f\|_{L_P[x_1 x]} \tag{21}$$

and

$$\left\| \int_{x_1}^{x_u} \big(g(t) - f(t)\big)dt \right\|_{L_P[x_1\,, x_u]} \le 6c_1c_2 \|g - f\|_{L_P[x_1, x_u]} \tag{22}$$

Then define g on the interval $[x_u\,, x_v]$, and $u < v \le n-1$, such that $g \in A_{u.,v}$, (21) remains effective , on $[x_1\,, x_v]$, and if $v < n-1$, then also

$$\left\| \int_{x_1}^{x_v} \big(g(t) - f(t)\big)dt \right\|_{L_P[x_1\,, x_v]} \le 6c_1c_2 \|g - f\|_{L_P[x_1\,,\, x_v]} \tag{23}$$

If

$$\int_{x_1}^{x_u} \big(g(t) - f(t)\big)dt \le 0 \tag{24}$$

Then put $v = u + 1$ and apply Lemma 2.2 on interval $[x_{v-1}, x_v]$, with $q = s_{\mid [x_{v-1}, x_v]}$. We take $(x) = p(x)$, and for each $x \in [x_{v-1}, x_v]$, where p is polynomial. For each $x \in [x_{v-1}, x_v]$, we have by (22) and (3) ,

$$\left\| \int_{x_1}^{x} \big(g(t) - f(t)\big)dt \right\|_{L_P[x_1\,, x]} \le \left\| \int_{x_1}^{x_u} \big(g(t) - f(t)\big)dt \right\|_{L_P[x_1\,, x_u]} + \left\| \int_{x_u}^{x} \big(g(t) - f(t)\big)dt \right\|_{L_P[x_u\,, x]}$$

Hence , the combining with (21) for each $x \in [x_1\,, x_u]$, we get that (21) holds for all $x \in [x_1\,, x_v]$.

Moreover , $\int_{x_u}^{x_v} p(t)dt \geq \int_{x_u}^{x_v} f(t)dt$ implies that

$$0 \leq \int_{x_{v-1}}^{x_v} \big(g(t) - f(t)\big)dt \ \leq 2c_1c_2\|g - f\|_{L_P[x_{v-1},x_v]}$$

Together with (22) and (24) , we obtain

$$-6c_1c_2\|g - f\|_{L_P[x_1\,,\,x_v]} \leq \int_{x_1}^{x_v} \big(g(t) - f(t)\big)dt \leq 2c_1c_2\|g - f\|_{L_P[x_1\,,\,x_v]}$$

This proves (23) . Note there is only place use of $\int_{x_u}^{x_v} p(t)dt \geq \int_{x_u}^{x_v} f(t)dt$

Otherwise **,**

$$\int_{x_1}^{x_u}\big(g(t) - f(t)\big)dt > 0 \tag{25}$$

By Lemma 2.12 , and use some integer v , $u + 1 < v \leq n - 1$, such that $g_{u,v}^* \in A_{u,v}$,by using (17) and (18) if $v < n - 1$. Take $(x) = g_{u,v}^*(x)$, for each $x \in [x_u, x_v]$. If $= n - 1$, then (17) implies (21) for all $x \in [x_1\,, x_{n-1}]$, and the construction is complete . Otherwise , for all $x \in [x_u, x_v]$,and by (17) and (22),

$$\begin{aligned}
&\left\|\int_{x_1}^{x}\big(g(t) - f(t)\big)dt\right\|_{L_P[x_1\,,x]} \\
&\leq \left\|\int_{x_1}^{x_u}\big(g(t) - f(t)\big)dt\right\|_{L_P[x_1\,,x_u]} + \left\|\int_{x_u}^{x}\big(g(t) - f(t)\big)dt\right\|_{L_P[x_u\,,x]} \\
&\leq 6c_1c_2\|g - f\|_{L_P} + 6c_1c_2\|g - f\|_{L_P} \\
&\leq 12c_1c_2\|g - f\|_{L_P[x_1\,,x]}
\end{aligned}$$

Hence , (21) holds for all $x \in [x_1\,, x_j]$. Also , by using (25) and (22), we get

$$0 \leq \int_{x_1}^{x_u} \big(g(t) - f(t)\big)dt \ \leq 6c_1c_2\|g - f\|_{L_P}$$

And combined with (17) and (18) give

$$-6c_1c_2\|g - f\|_{L_P} \leq \int_{x_1}^{x_v} \big(g(t) - f(t)\big)dt \leq 6c_1c_2\|g - f\|_{L_P}$$

This proves (23) and completes the induction.

Finally , in using of the definition of s , (21) , (2) holds. ■

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