

# Piecewise 3-Monotone Approximation on $L_P$ -Spaces for $P < 1$

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## Abstract

In this paper we introduce a Jackson type theorem for piecewise 3-monotone approximation of function in  $L_P$ -spaces for  $P < 1$ .

**Keywords:** 3-Monotone approximation by piecewise polynomials; Degree of approximation

## الخلاصة

قدمنا في هذا البحث نوعا من ميرهنه جاكسون للتقريب المتقطع للدوال رتيبة-3 للفضاءات  $L_P$  ,  $P < 1$ .  
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## 1.Introduction

Let  $f: [a, b] \rightarrow R$  is called  $K$ -monotone ,  $K \geq 1$  on  $[a, b]$  , if  $f[x_0, x_1, x_2, \dots, x_K] \geq 0$  for each choices of  $K + 1$  distinct point  $x_0, x_1, x_2, \dots, x_K$  in  $[a, b]$ .

Denote by

$$f[x_0, \dots, x_K] = \sum_{u=0}^K \frac{f(x_u)}{\prod_{v=0, v \neq u}^K (x_u - x_v)}$$

denotes the  $K$ th divide difference of  $f$  at the distinct point  $x_0, x_1, x_2, \dots, x_K$ . We denote that  $\Delta_{[a,b]}^1$  is nondecreasing function on  $[a, b]$  and  $\Delta_{[a,b]}^2$  is convex function on  $[a, b]$ . Note that  $\Delta_{[a,b]}^3$  is 3-monotone on  $[a, b]$ .

Higher order shape preserving approximation ,i.e  $K$ -monotone approximation , had been investigated in the results, many positive and negative results for  $K$ -monotone approximation  $K > 2$  , had been introduce which goes back to (Shvedov, 1981), the results "if  $f \in \Delta_{[-1,1]}^K \cap L_P^{(K)}$  , and  $0 \leq f^{(K)}(x) \leq 1$  ,  $\forall x \in [-1,1]$  , there exist apiecewise polynomial  $s \in \Delta_{[-1,1]}^K$  of degree  $\leq K - 1$  with distant  $n$  such that  $\|f(x) - s(x)\|_{L_P[-1,1]} \leq c(K)n^{-K}$  ,  $x \in [-1,1]$  ".

If  $F \in \Delta_{[a,b]}^3 \cap L_P^{(2)}$  Konovalov and Leviatan [Konovalov & Leviatan, 2001] have constructed 3-monotone quadratic spline  $S$  satisfying

$$\|F - S\|_{L_P[a,b]} \leq \frac{c}{n^2} \omega_1(F''; \frac{1}{n})$$

Where  $c = c(a, b)$  is absolute constant independent of  $F$  and .

In [Shvedov, 1981] Prymak extended the theorem of [Konovalov & Leviatan 2003; Leviatan& Prymak, 2005] and gave on estimate for the degree of approximation in terms

of the third-modulus of smoothness . As a consequence of the above result is the following for all  $F \in \Delta_{[a,b]}^3 \cap L_P[a, b]$  there is a piecewise quadratic  $S \in \Delta_{[a,b]}^3 \cap L_P[a, b]$  and with equidistant knot  $n$  . which

$$\|F - S\|_{L_P[a,b]} \leq c\omega_3(F; \frac{1}{n})$$

since  $c = c(a, b)$  is absolute constant .

The main aim of our paper is to answer the following question: Can we achieve a higher order estimate for the degree of best approximation of function  $s$  in  $L_P$  ,  $P < 1$  by 3-monotone piecewise polynomials?

In this paper we generalize the results in [Beatson, 1981; Bondarenko, 2002] to functions in  $L_P$ -spaces ,  $P < 1$  , i.e we prove

**Theorem 3.1 :** let  $F$  be a 3-monotone function in  $L_P[a, b]$  ,  $P < 1$  , satisfying  $f(x) = F'(x)$  , for each  $x \in (a, b)$  , and let  $k \geq 1$  ,  $a = x_0 < x_1 < \dots < x_n = b$  , be a partition for the interval  $[a, b]$ . Also let  $s$  be convex piecewise polynomial on  $[a, b]$  of degree  $\leq k - 1$  interpolating  $f$  at knots  $x_u$  ,  $u = 1, 2, \dots, n - 1$  , such that

$$s(x_u) = f(x_u) \tag{1}$$

Then there exist a 3-monotone polynomial in  $L_P[a, b]$  ,  $P$  of degree  $\leq k$  with the same knots and satisfy

$$\|F - P\|_{L_P[a,b]} \leq c(p)\|f - s\|_{L_P[x_{u-1}, x_u]} \tag{2}$$

$c(p)$  is an absolute constant depending on  $p$  only.

**Remark 1.1** [Leviatan& Prymak, 2005] :  $\Delta_{[a,b]}^3$  is the set of all bounded function, having a convex derivative on  $(a, b)$ . Note that if  $f \in \Delta_{[a,b]}^k$  ,  $k \geq 2$  , then  $f$  is continuous on  $(a, b)$  and  $f(a+)$  ,  $f(b-)$  exist and are finite.

**Defintion1.2**[Leviatan& Prymak, 2005]:let  $f, g$  real functions in  $L_P[a, b]$  . Then

$$\|f - g\|_{L_P[a,b]} \leq c(p)(\|f\|_{L_P[a,b]} + \|g\|_{L_P[a,b]})$$

and  $\|\cdot\|_{L_P[a,b]}$  is called quasinormed space  $L_P$  ,  $P < 1$  and  $c(p)$  is an absolute constant depending on  $P$  .

## 2. The auxiliary result

In this section we introduce the lemmas that we need to prove our theorem.

**Lemma 2.1**[Leviatan& Prymak, 2005]: let  $f \in \Delta_{[a,b]}^2$  and suppose that  $q \in \Delta_{[a,b]}^2$  is a polynomial of degree  $\leq k - 1$  satisfying  $f(a) = q(a)$  and  $f(b) = q(b)$  . Then there exists a polynomial  $p \in \Delta_{[a,b]}^2$  of degree  $\leq k - 1$  such that

$$(1) p(x) = \frac{A l(x) + B q(x)}{A + B} ,$$

Where  $A = \int_a^b f(t) dt - \int_a^b q(t) dt$  ,  $B = \int_a^b l(t) dt - \int_a^b f(t) dt$

and  $l$  be the Lagrange polynomial of  $f$  at  $a$  and  $b$ .

$$(2) f(a) = p(a), f(b) = p(b)$$

$$q'(a) \leq p', p'(b) \leq q'(b)$$

$$(3) \int_a^b p(x)dx \geq \int_a^b f(x)dx.$$

**Lemma 2.2:** let  $f \in \Delta_{[a,b]}^2 \cap L_p[a,b], p < 1$  and assume  $q$  is a convex polynomial of degree  $\leq k - 1$  in  $L_p[a,b]$  interpolation  $f$  at  $a$  and  $b$ . Then there is a convex polynomial  $p$  in  $L_p[a,b]$  of degree  $\leq k - 1$ , satisfying

(1) interpolation  $f$  at  $a$  and  $b$ .

(2)  $q'(a) \leq p'(a), p'(b) \leq q'(b)$ .

$$(3) \left\| \int_a^{(\cdot)} (p(t) - f(t))dt \right\|_{L_p[a,b]} \leq c(p) \left\| \int_a^{(\cdot)} (q(t) - f(t))dt \right\|_{L_p[a,b]}$$

$$(4) \int_a^b p(t)dt \geq \int_a^b f(t)dt.$$

**Proof:** by Lemma 2.1 we get (1), (2) and (4)

And by Lemma 2.1 ,

$$\text{Let } B = \int_a^b l(t)dt - \int_a^b f(t)dt ,$$

$$A = \int_a^b f(t)dt - \int_a^b q(t)dt$$

and

$$p(x) = \frac{Al(x) + Bq(x)}{A + B}$$

To prove (3),

$$\begin{aligned} \left\| \int_a^x p(t)dt - \int_a^x f(t)dt \right\|_{L_p[a,b]} &= \left\| \int_a^x \left( \frac{Al(t) - Bq(t)}{A + B} \right) dt - \int_a^x f(t)dt \right\|_{L_p[a,b]} \\ &= \left\| \frac{A}{A + B} \int_a^x (l(t) - f(t))dt + \frac{A}{A + B} \int_a^x (q(t) - f(t))dt \right\|_{L_p[a,b]} \\ &\leq c(p) \left( \frac{A}{A + B} \left\| \int_a^x (l(t) - f(t))dt \right\|_{L_p[a,b]} + \frac{B}{A + B} \left\| \int_a^x (q(t) - f(t))dt \right\|_{L_p[a,b]} \right) \\ &\leq c(p) \left( \frac{A}{A + B} B + \frac{B}{A + B} \left\| \int_a^{(\cdot)} (q(t) - f(t))dt \right\|_{L_p[a,b]} \right) \\ &\leq c(p) \frac{B}{A + B} \left\| \int_a^{(\cdot)} (q(t) - f(t))dt \right\|_{L_p[a,b]} + \frac{B}{A + B} \left\| \int_a^{(\cdot)} (q(t) - f(t))dt \right\|_{L_p[a,b]} \end{aligned}$$

$$\leq c(p) \left\| \int_a^{(\cdot)} (q(t) - f(t)) dt \right\|_{L_p[a,b]} \quad \blacksquare \quad (3)$$

**Lemma 2.3** [Leviatan& Prymak, 2005]:let  $q \in \Delta_{[a,b]}^2$  be polynomial of degree  $\leq n - 1$  , and let  $\alpha$  and  $\beta$  arbitrary non-negative real numbers . Suppose that  $d_a, d_b$  are real numbers satisfying ,

$$d_a \leq \frac{(q(b)-\beta)-(q(a)-\alpha)}{b-a} \leq d_b \quad (4)$$

And  $d_a \leq q'(a) \leq q'(b) \leq d_a$  . Then there exists a polynomial  $p \in \Delta_{[a,b]}^2$  of degree  $\leq n - 1$  , such that

$$p(a) = q(a) - \alpha , p(b) = q(b) - \beta \quad (5)$$

$$d_a \leq p'(a) \leq p'(b) \leq d_a \quad (6)$$

and

$$p(x) \leq q(x) , x \in [a, b] \quad (7)$$

**Lemma 2.4:** let  $f, g \in \Delta_{[z_1, z_2]}^2 \cap L_p[z_1, z_2]$  ,  $p < 1$  , satisfying  $f(z_u) = g(z_u)$  and  $l_u(x) = (x - z_u)g'(z_u) + g(z_u)$  ,  $u = 1, 2$

Then

$$\int_{z_1}^{z_2} (l_u(t) - f(t)) dt \leq c_1 c_2 \|g(t) - f(t)\|_{L_p[z_1, z_2]}$$

**Proof:** if  $u = 1$  , let  $g$  is convex , and  $l_1 \leq g(x)$  , for each  $x \in [z_1, z_2]$ . Let  $f$  is convex and  $l_u$  is linear , there is a  $\theta \in [z_1, z_2]$  , so that  $f(x) \leq l_1(x)$  , for each  $x \in [z_1, \theta]$ , such that  $l_1(x) = f(x)$  , for each  $x \in [\theta, z_2]$  .

if  $u = 1$

$$\begin{aligned} \int_{z_1}^{\theta} (g(t) - f(t)) dt &= \left( \int_{z_1}^{\theta} (g(t) - f(t))^{1-p+p} dt \right)^{1-\frac{1}{p} + \frac{1}{p}} \\ &= \left( \int_{z_1}^{\theta} (g(t) - f(t))^{1-p} (g(t) - f(t))^p dt \right)^{\frac{1}{p}} \left( \int_{z_1}^{\theta} (g(t) - f(t))^{1-p} (g(t) - f(t))^p dt \right)^{1-\frac{1}{p}} \end{aligned}$$

$$\leq c_1 c_2 \left( \int_{z_1}^{\theta} (g(t) - f(t))^p dt \right)^{\frac{1}{p}} ,$$

Where  $c_1 = \left( \int_{z_1}^{\theta} (g(t) - f(t))^{1-p} dt \right)^{\frac{1}{p}}$  ,

and  $c_2 = \left( \int_{z_1}^{\theta} (g(t) - f(t))^{1-p} (g(t) - f(t))^p dt \right)^{1-\frac{1}{p}}$

$$\int_{z_1}^{\theta} (g(t) - f(t))dt \leq c_1 c_2 \|g - f\|_{L_P[z_1, \theta]}$$

if  $u = 2$

$$\begin{aligned} \int_{\theta}^{z_2} (g(t) - f(t))dt &= \left( \int_{\theta}^{z_2} (g(t) - f(t))^{1-p+p} dt \right)^{1-\frac{1}{p} + \frac{1}{p}} \\ &= \left( \int_{\theta}^{z_2} (g(t) - f(t))^{1-p} (g(t) - f(t))^p dt \right)^{\frac{1}{p}} \left( \int_{\theta}^{z_2} (g(t) - f(t))^{1-p} (g(t) - f(t))^p dt \right)^{1-\frac{1}{p}} \\ &\leq c_1 c_2 \left( \int_{\theta}^{z_2} (g(t) - f(t))^p dt \right)^{\frac{1}{p}}, \end{aligned}$$

where  $c_1 = \left( \int_{\theta}^{z_2} (g(t) - f(t))^{1-p} dt \right)^{\frac{1}{p}}$ ,

and

$$c_2 = \left( \int_{\theta}^{z_2} (g(t) - f(t))^{1-p} (g(t) - f(t))^p dt \right)^{1-\frac{1}{p}}$$

$$\int_{\theta}^{z_2} (g(t) - f(t))dt \leq c_1 c_2 \|g - f\|_{L_P[\theta, z_2]}$$

Hence ,

$$\int_{z_1}^{z_2} (l_u(t) - f(t))_+ dt \leq c_1 c_2 \|g(t) - f(t)\|_{L_P[z_1, z_2]} \quad \blacksquare$$

**Lemma 2.5** [Leviatan& Prymak, 2005]: let  $f, g \in \Delta_{[a,b]}^2$  be such that

$$f(b) - f(a) = g(b) - g(a) \tag{8}$$

Then

$$f'(a+) \leq g'(b-)$$

**Corollary 2.6** [Leviatan& Prymak, 2005]: let  $f \in \Delta_{[a,b]}^2$  and  $s \in \Delta_{[a,b]}^2$  be a piecewise polynomial of degree  $\leq n - 1$  , with knots on the partition  $a = x_0 < x_1 < \dots < x_n = b$  , satisfying

$s(x_u) = f(x_u)$  . Then for  $u = 2, \dots, n - 1$

$$f'(x_{u-1}+) \leq s'(x_u-) , \quad u = 2, \dots, n - 1 \tag{9}$$

and

$$s'(x_{u-1}+) \leq f'(x_u-) , \quad u = 2, \dots, n - 1 \tag{10}$$

**Remark 2.7:** we are begin our auxiliary construction . Let  $f, s$  are convex a piecewise polynomial on  $[a, b]$  of degree  $\leq n - 1$  . Then the function  $g$  , we write  $g \in A_{u,v}$  ,  $u \leq v \leq n - 1$  , and  $g$  is convex a piecewise polynomial on  $[x_u, x_v]$  of degree  $\leq n - 1$  , with knots  $x_{u+1}, \dots, x_{v-1}$  , and satisfies  $s'(x_u +) \leq g'(x_u +)$  and  $g'(x_v -) \leq s'(x_v -)$  and  $g(x_u) = s(x_u)$  and  $g(x_v) = s(x_v)$  ,  $\forall r = 1, \dots, n - 1$  . [2]

Let

$$h_r(t) = \begin{cases} f'(x_u -) & IF \ t \in (x_{u-1}, x_u] , & u = 1, 2, \dots, n - 1 \\ s'(x_r -) & IF \ t \in (x_{r-1}, x_u] , & u = 1, 2, \dots, n - 1 \\ s'(x_r +) & IF \ t \in (x_r, x_{r+1}) , & u = 1, 2, \dots, n - 1 \\ f'(x_{i-1} +) & IF \ t \in [x_{u-1}, x_u) , & u = 1, 2, \dots, n - 1 \end{cases} \quad [2]$$

and

$$\text{let } g_r(x) = f(x_r) + \int_{x_r}^x h_r(t)dt \quad [\text{Leviatan\& Prymak, 2005}]$$

By virtue of (Corollary 2.6 ) ,  $h_r$  is non-decreasing on  $(a,b)$  , and  $g_r$  is convex . It follow by  $s(x_u) = f(x_u)$  that  $g_r(x_{r+1}) \leq f(x_{r+1})$  and  $g_r(x) \leq f(x_{r-1})$  .Hence,

$$g_r(x) \leq f(x) , \quad x \in [a, b] \setminus (x_{r-1}, x_{r+1}) \quad [\text{Leviatan\& Prymak, 2005}] \quad (11)$$

By Lemma 2.4,

$$\begin{aligned} \int_{x_r}^{x_{r+1}} (g_r(t) - f(t))dt &= \left( \int_{x_r}^{x_{r+1}} (g_r(t) - f(t))^{1-p+p} dt \right)^{1-\frac{1}{p} + \frac{1}{p}} \\ &= \left( \int_{x_r}^{x_{r+1}} (g_r(t) - f(t))^{1-p} (g_r(t) - f(t))^p dt \right)^{\frac{1}{p}} \left( \int_{x_r}^{x_{r+1}} (g_r(t) - f(t))^{1-p} (g_r(t) \right. \\ &\quad \left. - f(t))^p dt \right)^{1-\frac{1}{p}} \end{aligned}$$

$$\leq c_1 c_2 \left( \int_{x_r}^{x_{r+1}} (g_r(t) - f(t))^p dt \right)^{\frac{1}{p}} ,$$

$$\text{where } c_1 = \left( \int_{x_r}^{x_{r+1}} (g_r(t) - f(t))^{1-p} dt \right)^{\frac{1}{p}}$$

and

$$c_2 = \left( \int_{x_r}^{x_{r+1}} (g_r(t) - f(t))^{1-p} (g_r(t) - f(t))^p dt \right)^{1-\frac{1}{p}}$$

$$\int_{x_r}^{x_{r+1}} (g_r(t) - f(t))dt \leq c_1 c_2 \|g - f\|_{L^p[x_r, x_{r+1}]} \quad (12)$$

And

$$\int_{x_{r-1}}^{x_r} (g_r(t) - f(t))dt = \left( \int_{x_{r-1}}^{x_r} (g_r(t) - f(t))^{1-p+p} dt \right)^{1-\frac{1}{p} + \frac{1}{p}}$$

$$= \left( \int_{x_{r-1}}^{x_r} (g_r(t) - f(t))^{1-p} (g(t) - f(t))^p \right)^{\frac{1}{p}} \left( \int_{x_{r-1}}^{x_r} (g_r(t) - f(t))^{1-p} (g_r(t) - f(t))^p \right)^{1-\frac{1}{p}} dt$$

$$\leq c_1 c_2 \left( \int_{x_{r-1}}^{x_r} (g_r(t) - f(t))^p dt \right)^{\frac{1}{p}},$$

Where

$$c_1 = \left( \int_{x_{r-1}}^{x_r} (g_r(t) - f(t))^{1-p} \right)^{\frac{1}{p}}$$

and

$$c_2 = \left( \int_{x_{r-1}}^{x_r} (g_r(t) - f(t))^{1-p} (g_r(t) - f(t))^p \right)^{1-\frac{1}{p}}$$

Hence,

$$\int_{x_{r-1}}^{x_r} (g_r(t) - f(t)) dt \leq c_1 c_2 \|g - f\|_{L^p[x_{r-1}, x_r]} \tag{13}$$

**Lemma 2.8** [Leviatan .D., Prymak .A.V.]:

$$\int_{x_{m-1}}^{x_m} (g_{u,v}(t) - f(t))_+ dt \leq \int_{x_{m-1}}^{x_m} (s(t) - f(t))_+ dt, \text{ for each } x \in [x_{m-1}, x_m]$$

Where  $s \in \Delta_{[a,b]}^2$  be a piecewise polynomial of degree  $\leq n - 1$ .

**Remark 2.9:** by virtue of (12) ,(13) we have

$$\begin{aligned} \int_{x_u}^{x_{u+1}} (g_{u,v}(t) - f(t)) dt &= \left( \int_{x_u}^{x_{u+1}} (g_{u,v}(t) - f(t))^{1-p+p} dt \right)^{1-\frac{1}{p}+\frac{1}{p}} \\ &= \left( \int_{x_u}^{x_{u+1}} (g_{u,v}(t) - f(t))^{1-p} (g(t) - f(t))^p \right)^{\frac{1}{p}} \left( \int_{x_u}^{x_{u+1}} (g_{u,v}(t) - f(t))^{1-p} (g_{u,v}(t) - f(t))^p \right)^{1-\frac{1}{p}} dt \end{aligned}$$

$$\leq c_1 c_2 \left( \int_{x_v}^{x_{u+1}} (g_{u,v}(t) - f(t))^p dt \right)^{\frac{1}{p}},$$

$$\text{where } c_1 = \left( \int_{x_u}^{x_{u+1}} (g_{u,v}(t) - f(t))^{1-p} \right)^{\frac{1}{p}}$$

and

$$c_2 = \left( \int_{x_u}^{x_{u+1}} (g_{u,v}(t) - f(t))^{1-p} (g_{u,v}(t) - f(t))^p \right)^{1-\frac{1}{p}}$$

$$\int_{x_u}^{x_{u+1}} (g_{u,v}(t) - f(t)) dt \leq c_1 c_2 \|g - f\|_{L_P[x_u, x_{u+1}]} \quad (14)$$

And

$$\begin{aligned} \int_{x_{v-1}}^{x_v} (g_{u,v}(t) - f(t)) dt &= \left( \int_{x_{v-1}}^{x_v} (g_{u,v}(t) - f(t))^{1-p+p} dt \right)^{1-\frac{1}{p}+\frac{1}{p}} \\ &= \left( \int_{x_{v-1}}^{x_v} (g_{u,v}(t) - f(t))^{1-p} (g(t) - f(t))^p dt \right)^{\frac{1}{p}} \left( \int_{x_{v-1}}^{x_v} (g_{u,v}(t) - f(t))^{1-p} (g_{u,v}(t) - f(t))^p dt \right)^{1-\frac{1}{p}} \\ &\leq c_1 c_2 \left( \int_{x_{v-1}}^{x_v} (g_{u,v}(t) - f(t))^p dt \right)^{\frac{1}{p}}, \end{aligned}$$

where  $c_1 = \left( \int_{x_{v-1}}^{x_v} (g_{u,v}(t) - f(t))^{1-p} dt \right)^{\frac{1}{p}}$

and  $c_2 = \left( \int_{x_{v-1}}^{x_v} (g_{u,v}(t) - f(t))^{1-p} (g_{u,v}(t) - f(t))^p dt \right)^{1-\frac{1}{p}}$

$$\int_{x_{v-1}}^{x_v} (g_{u,v}(t) - f(t)) dt \leq c_1 c_2 \|g - f\|_{L_P[x_{v-1}, x_v]} \quad (15)$$

**Lemma 2.10 :** let  $g_{u,v}(x) \leq f(x)$  ,  $\forall x \in [x_{l-1}, x_l]$  , and  $u + 1 < l < v$  ,  $l \neq m$  and  $g_{u,v}$  is a convex polynomial of degree  $\leq k - 1$

Then

$$\int_{x_u}^{x_v} (g_{u,v}(t) - f(t)) dt \leq 3c_1 c_2 \|g - f\|_{L_P[x_u, x_v]}$$

**Proof:** from Lemma 2.8 ,(14)and(15) ,we have

$$\begin{aligned} \int_{x_u}^{x_v} (g_{u,v}(t) - f(t)) dt &\leq \int_{x_u}^{x_{u+1}} (g_{u,v}(t) - f(t)) dt + \int_{x_{m-1}}^{x_m} (g_{u,v}(t) - f(t)) dt \\ &\quad + \int_{x_{v-1}}^{x_v} (g_{u,v}(t) - f(t)) dt \end{aligned}$$

$$\leq c_1 c_2 \|g - f\|_{L_P[x_u, x_v]} + c_1 c_2 \|g - f\|_{L_P[x_u, x_{u+1}]} + c_1 c_2 \|g - f\|_{L_P[x_v, x_{v-1}]}$$

Where  $c_1$  and  $c_2$  as we define above.

$$\int_{x_u}^{x_v} (g_{u,v}(t) - f(t)) dt \leq 3c_1 c_2 \|g - f\|_{L_P[x_u, x_v]} \quad \blacksquare$$



**Lemma 2.11:** let  $\delta(\cdot) \in L_P[a, b], P < 1$  ,then we have

$$\left\| \int_{x_u}^{(\cdot)} \delta(t) dt \right\|_{L_P[x_u, x_v]} \leq \left\| \int_{x_u}^{x_v} \delta(t) dt \right\|_{L_P[x_u, x_v]} + \left\| \int_{x_u}^{x_v} \delta(t)_+ dt \right\|_{L_P[x_u, x_v]}$$

**Proof:** let  $x_u \leq x \leq x_v$  if  $\int_{x_u}^x \delta(t) dt \geq 0$  , then

$$0 \leq \left\| \int_{x_u}^x \delta(t)_+ dt \right\|_{L_P[x_u, x]} \leq \left\| \int_{x_u}^x \delta_+(t) dt \right\|_{L_P[x_u, x]} \leq \left\| \int_{x_u}^{x_v} \delta_+(t) dt \right\|_{L_P[x_u, x_v]}$$

And if  $\int_{x_u}^x \delta(t) < 0$  ,

$$\begin{aligned} \left\| \int_{x_u}^x \delta(t) dt \right\|_{L_P[x_u, x]} &\leq \left\| \int_{x_u}^x \delta_-(t) dt \right\|_{L_P[x_u, x]} \leq \left\| \int_{x_u}^{x_v} \delta_-(t) dt \right\|_{L_P[x_u, x_v]} \\ &\leq \left\| - \int_{x_u}^{x_v} \delta(t) dt \right\|_{L_P[x_u, x_v]} + \left\| \int_{x_u}^{x_v} \delta(t) dt \right\|_{L_P[x_u, x_v]} \end{aligned}$$

Hence  $\left\| \int_{x_u}^{(\cdot)} \delta(t) dt \right\|_{L_P[x_u, x_v]} \leq \left\| \int_{x_u}^{x_v} \delta(t) dt \right\|_{L_P[x_u, x_v]} + \left\| \int_{x_u}^{x_v} \delta_+(t) dt \right\|_{L_P[x_u, x_v]}$

Then by Lemma 2.10 , we have

$$\left\| \int_{x_u}^{(\cdot)} (g_{u,v}(t) - f(t)) dt \right\|_{L_P[x_u, x]} \leq \left\| \int_{x_u}^{x_v} (g_{u,v}(t) - f(t)) dt \right\|_{L_P[x_u, x_v]} + 3c_1c_2 \|g - f\|_{L_P[x_u, x_v]} \quad (16)$$

**Lemma 2.12 :** The fixed integer  $1 \leq u \leq n - 2$  . Then there is  $u + 1 \leq v \leq n - 1$  , and afunction  $g_{u,v}^* \in A_{u,v}$  and  $A_{u,v}$  is a convex piecewise polynomial of degree  $\leq k - 1$

such that

$$\left\| \int_{x_u}^{(\cdot)} (g_{u,v}^*(t) - f(t)) dt \right\|_{L_P} \leq 6c_1c_2 \|g - f\|_{L_P} \quad (17)$$

If  $v \leq n - 1$  , then

$$\int_{x_u}^{x_v} (g_{u,v}^*(t) - f(t)) dt \leq 0 \quad (18)$$

**Proof :** If  $\int_{x_u}^{x_{n-1}} (g_{u,v}(t) - f(t)) dt \geq 0$

Then by Lemma 2.10 ,

$$\int_{x_u}^{x_{n-1}} (g_{u,v}(t) - f(t)) dt \leq 3c_1c_2 \|g - f\|_{L_P[x_u, x_v]}$$

and put  $g_{u,n-1}^* = g_{u,n-1}$

(17) follows by (16)

Otherwise, at smallest of the above numbers  $\int_{x_u}^{x_{u+r}} (g_{u,v}(t) - f(t))dt$  ,  $u \leq r \leq n - 1$  is anegative .

$$\text{Let } u \leq r \leq n - u - 1 , -6c_1c_2\|g - f\|_{L_P[x_u, x_v]} \leq \int_{x_u}^{x_{u+r}} (g_{u,v}(t) - f(t)) dt < 0$$

We put  $v = u + r$  so that  $g_{u,v}^* = g_{u,v}$  ,then (18) is complete , and by (16) ,We get (17).

Finally,if all numbers are anegative among the above are  $< -6c_1c_2\|g - f\|_{L_P}$  , then let  $1 \leq r \leq n - u - 1$  ,be the little such that

$$\left\| \int_{x_u}^{x_{u+r}} (g_{u,v}(t) - f(t)) dt \right\|_{L_P} \leq -6c_1c_2\|g - f\|_{L_P} , r \geq 2$$

since  $g_{u,u+1}(x) = s(x)$  ,for all  $x \in [x_u , x_{v+1}]$  ,

$$\text{whence } \left\| \int_{x_u}^{x_{u+1}} (g_{u,v}(t) - f(t)) dt \right\|_{L_P} \leq c_1c_2\|g - f\|_{L_P} ,$$

Put  $v = u + r$  , and  $p = s|_{[x_{v-1}, x_v]}$ ,

Then

$$\left\| \int_{x_{v-1}}^{(\cdot)} (p(t) - f(t))dt \right\|_{L_P[x_{v-1}, x_v]} \leq c_1c_2\|g - f\|_{L_P} \tag{19}$$

Let

$$\ddot{g}_{u,v}(x) = \begin{cases} g_{u,v-1}(x) & IF \quad x \in [x_u , x_{v-1}] \\ p(x) & IF \quad x \in [x_{v-1} , x_v] \end{cases}$$

Then  $\ddot{g}_{u,v} \in A_{u,v-1}$  and  $\ddot{g}_{u,v} \in A_{v-1,v}$  , such that  $\ddot{g}_{u,v} \in A_{u,v}$  . Let  $g_{u,v}^*(x) = \lambda g_{u,v}(x) + (1 - \lambda)\ddot{g}_{u,v}(x)$  , for all  $x \in [x_u , x_v]$  ,

$$\text{where } \lambda = 6c_1c_2\|g - f\|_{L_P} / \left\| \int_{x_u}^{x_v} (g_{u,v}(t) - f(t)) dt \right\|_{L_P} = 5 c_1c_2\|g - f\|_{L_P} , \lambda > 0$$

So that  $g_{u,v}^* \in A_{u,v}$  .The select  $r$  implies that  $0 \leq \int_{x_u}^{x_{v-1}} (g_{u,v}(t) - f(t)) dt \leq 3c_1c_2\|g - f\|_{L_P}$

then by (16)

$$\begin{aligned} \left\| \int_{x_u}^{(\cdot)} (\ddot{g}_{u,v}(t) - f(t))dt \right\|_{L_P[x_u, x]} &= \left\| \int_{x_u}^{x_{v-1}} (\ddot{g}_{u,v}(t) - f(t))dt \right\|_{L_P[x_u, x_{v-1}]} \\ &\leq \left\| \int_{x_u}^{x_{v-1}} (\ddot{g}_{u,v}(t) - f(t))dt \right\|_{L_P} + 3c_1c_2\|g - f\|_{L_P} \\ &\leq 3c_1c_2\|g - f\|_{L_P} + 3c_1c_2\|g - f\|_{L_P} \\ &\leq 6c_1c_2\|g - f\|_{L_P} \end{aligned}$$

Also ,by (19)

$$\begin{aligned} & \left\| \int_{x_u}^x (\ddot{g}_{u,v}(t) - f(t))dt \right\|_{L_P} = \left\| \int_{x_u}^{x_{v-1}} (\ddot{g}_{u,v}(t) - p(t)) dt + \int_{x_{v-1}}^x (p(t) - f(t))dt \right\|_{L_P} \\ & \leq 2^{p-1} \left( \left\| \int_{x_u}^{x_{v-1}} (\ddot{g}_{u,v}(t) - f(t)) dt \right\|_{L_P} + \left\| \int_{x_{v-1}}^x (p(t) - f(t))dt \right\|_{L_P} \right) \\ & \leq 2^{p-1} (3c_1c_2 \|g - f\|_{L_P} + c_1c_2 \|g - f\|_{L_P}) \\ & \leq 4(2^{p-1}c_1c_2 \|g - f\|_{L_P[x_{v-1},x_v]}) \quad , \quad x \in [x_{v-1}, x_u] \\ & \leq 2^{p+1}c_1c_2 \|g - f\|_{L_P} \end{aligned}$$

Therefore ,

$$\left\| \int_{x_u}^{(\cdot)} (\ddot{g}_{u,v}(t) - f(t))dt \right\|_{L_P[x_u,x_v]} \leq 6c_1c_2 \|g - f\|_{L_P} \tag{20}$$

In particular

$$\begin{aligned} & \int_{x_u}^{x_v} (g_{u,v}(t) - f(t)) dt \\ & = \lambda \int_{x_u}^{x_v} (g_{u,v}(t) - f(t)) dt + (1 - \lambda) \int_{x_u}^{x_v} (\ddot{g}_{u,v}(t) - f(t)) dt \\ & \leq 5(-6c_1c_2 \|g - f\|_{L_P}) + (1 - 5)6c_1c_2 \|g - f\|_{L_P} < 0 \end{aligned}$$

We get (18) .Then by (16)and (20) ,

$$\begin{aligned} & \left\| \int_{x_u}^{x_v} (g_{u,v}(t) - f(t)) dt \right\|_{I_p[x_u,x_v]} \\ & \leq \lambda \left\| \int_{x_u}^{x_v} (g_{u,v}(t) - f(t)) dt \right\|_{I_p[x_u,x_v]} + (1 \\ & \quad - \lambda) \left\| \int_{x_i}^{(\cdot)} (\ddot{g}_{u,v}(t) - f(t))dt \right\|_{I_p[x_u,x_v]} \\ & \leq \lambda \left\| \int_{x_u}^{x_v} (g_{u,v}(t) - f(t)) dt \right\|_{L_P[x_u,x_v]} + 3c_1c_2 \|g - f\|_{L_P} + (1 - \lambda)6c_1c_2 \|g - f\|_{L_P} \\ & \leq 5(6c_1c_2 \|g - f\|_{L_P}) + 6(1 - 5)c_1c_2 \|g - f\|_{L_P} \\ & \leq 6c_1c_2 \|g - f\|_{L_P} \quad \text{this proves (17)} \quad \blacksquare \end{aligned}$$

### 3. The main result

In this section we introduce our main result

**3.1 :The proof of theorem.**

For the proof we follow the same strategies used in [Leviatan& Prymak, 2005] .

We search for the require function  $P$  in the

$$P(x) = F(x_u) + \int_{x_1}^x \hat{g}(t)dt \quad , \quad \forall x \in [a, b]$$

Let

$$\hat{g}(t) = \begin{cases} s(t) & \text{if } t \in [x_0, x_1) \cup (x_{n-1}, x_n] \\ g(t) & \text{if } t \in [x_1, x_{n-1}] \end{cases}$$

In  $A_{1,n-1}$  . The last we shall construct the 3-monotone polynomial of  $P$  .Construct  $g$  by induction.

Apply Lemma 2.2 on interval  $[x_{u-1}, x_u]$  ,  $2 \leq u \leq n - 1$  , and  $q = s|_{[x_{u-1}, x_u]}$  , then the polynomial  $p$  is in  $A_{u-1,u}$  . Also, recall if  $g \in A_{u,v}$  ,  $1 \leq u < v < l \leq n - 1$  , and  $g \in A_{u,v}$  , such that  $g \in A_{u,l}$  . We get construct  $g$  by induction. Apply Lemma 2.2 on interval  $[x_1, x_2]$  , with  $q = s|_{[x_1, x_2]}$  we obtain on apolynomial  $p \in A_{1,2}$  , and take  $p(x) = g(x)$  . Assume that  $g$  is define on  $[x_1, x_2]$  ,  $1 + 1 \leq u \leq n - 1 - 1$  ,  $g \in A_{1,u}$  , and satisfying for each  $x \in [x_1, x_u]$  ,

$$\left\| \int_{x_1}^x (g(t) - f(t))dt \right\|_{L_P[x_1, x]} \leq 12c_1c_2 \|g - f\|_{L_P[x_1, x]} \quad (21)$$

and

$$\left\| \int_{x_1}^{x_u} (g(t) - f(t))dt \right\|_{L_P[x_1, x_u]} \leq 6c_1c_2 \|g - f\|_{L_P[x_1, x_u]} \quad (22)$$

Then define  $g$  on the interval  $[x_u, x_v]$  , and  $u < v \leq n - 1$  , such that  $g \in A_{u,v}$  , (21) remains effective , on  $[x_1, x_v]$  , and if  $v < n - 1$  , then also

$$\left\| \int_{x_1}^{x_v} (g(t) - f(t))dt \right\|_{L_P[x_1, x_v]} \leq 6c_1c_2 \|g - f\|_{L_P[x_1, x_v]} \quad (23)$$

If

$$\int_{x_1}^{x_u} (g(t) - f(t))dt \leq 0 \quad (24)$$

Then put  $v = u + 1$  and apply Lemma 2.2 on interval  $[x_{v-1}, x_v]$  , with  $q = s|_{[x_{v-1}, x_v]}$  . We take  $(x) = p(x)$  , and for each  $x \in [x_{v-1}, x_v]$  , where  $p$  is polynomial. For each  $x \in [x_{v-1}, x_v]$  , we have by (22) and (3) ,

$$\left\| \int_{x_1}^x (g(t) - f(t))dt \right\|_{L_P[x_1, x]} \leq \left\| \int_{x_1}^{x_u} (g(t) - f(t))dt \right\|_{L_P[x_1, x_u]} + \left\| \int_{x_u}^x (g(t) - f(t))dt \right\|_{L_P[x_u, x]}$$

Hence , the combining with (21) for each  $x \in [x_1, x_u]$  , we get that (21) holds for all  $x \in [x_1, x_v]$ .

Moreover ,  $\int_{x_u}^{x_v} p(t)dt \geq \int_{x_u}^{x_v} f(t)dt$  implies that

$$0 \leq \int_{x_{v-1}}^{x_v} (g(t) - f(t))dt \leq 2c_1c_2\|g - f\|_{L_P[x_{v-1}, x_v]}$$

Together with (22) and (24) , we obtain

$$-6c_1c_2\|g - f\|_{L_P[x_1, x_v]} \leq \int_{x_1}^{x_v} (g(t) - f(t))dt \leq 2c_1c_2\|g - f\|_{L_P[x_1, x_v]}$$

This proves (23) . Note there is only place use of  $\int_{x_u}^{x_v} p(t)dt \geq \int_{x_u}^{x_v} f(t)dt$

Otherwise ,

$$\int_{x_1}^{x_u} (g(t) - f(t))dt > 0 \tag{25}$$

By Lemma 2.12 , and use some integer  $v$  ,  $u + 1 < v \leq n - 1$  , such that  $g_{u,v}^* \in A_{u,v}$  ,by using (17) and (18) if  $v < n - 1$  . Take  $(x) = g_{u,v}^*(x)$  , for each  $x \in [x_u, x_v]$  . If  $= n - 1$  , then (17) implies (21) for all  $x \in [x_1, x_{n-1}]$  , and the construction is complete . Otherwise , for all  $x \in [x_u, x_v]$  ,and by (17) and (22),

$$\begin{aligned} & \left\| \int_{x_1}^x (g(t) - f(t))dt \right\|_{L_P[x_1, x]} \\ & \leq \left\| \int_{x_1}^{x_u} (g(t) - f(t))dt \right\|_{L_P[x_1, x_u]} + \left\| \int_{x_u}^x (g(t) - f(t))dt \right\|_{L_P[x_u, x]} \\ & \leq 6c_1c_2\|g - f\|_{L_P} + 6c_1c_2\|g - f\|_{L_P} \\ & \leq 12c_1c_2\|g - f\|_{L_P[x_1, x]} \end{aligned}$$

Hence , (21) holds for all  $x \in [x_1, x_j]$  . Also , by using (25) and (22), we get

$$0 \leq \int_{x_1}^{x_u} (g(t) - f(t))dt \leq 6c_1c_2\|g - f\|_{L_P}$$

And combined with (17) and (18) give

$$-6c_1c_2\|g - f\|_{L_P} \leq \int_{x_1}^{x_v} (g(t) - f(t))dt \leq 6c_1c_2\|g - f\|_{L_P}$$

This proves (23) and completes the induction.

Finally , in using of the definition of  $s$  , (21) , (2) holds. ■

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